On Solutions of Evolution Equations with Proportional Time Delay

Weijiu Liu* and John C. Clements†
Department of Mathematics and Statistics
Dalhousie University
Halifax, Nova Scotia, B3H 3J5, Canada
Fax: (902) 494-5130
E-mail: weiliu@mathstat.dal.ca and clements@mathstat.dal.ca

Abstract

This work investigates reaction-diffusion equations of the form $y_t(x,t) = \kappa\Delta y(x,t) + ay(x, \lambda t - \sigma)$ which incorporate a proportional or general linear time delay $\lambda t - \sigma$ where $\lambda, \sigma \geq 0$, $\kappa > 0$ and a is a possibly complex constant. We establish the existence of unique solutions on $[0,T]$ for any T if $0 < \lambda \leq 1$ and for $0 < T < \sigma/(\lambda - 1)$ if $\lambda > 1$ and $\sigma > 0$. It is shown that if $0 < \lambda < 1$, then there exists a constant μ_0 such that these solutions do not grow faster than a polynomial of degree $p = \frac{1}{\ln \lambda} \ln\left(\frac{\kappa\mu_0}{|a|}\right)$ and, moreover, if $|a| < \kappa\mu_0$, then $p < 0$ and the solutions decay to zero at polynomial rate as $t \to \infty$. Similar results are obtained for equations involving power time delay terms of the form $ay(x, t^\lambda)$.

AMS Subject Classification. 35R10, 35B35, 35K05.
Keywords. proportional delay, evolution equations, asymptotic behavior.

---

*Supported by the Killam Postdoctoral Fellowship.
†Supported by NSERC of Canada.

1 Introduction

Let a be a possibly complex constant and b and λ real constants with $\lambda \geq 0$. Functional-differential equations with proportional delay of the form

$$y'(t) = ay(\lambda t) + by(t) \qquad (0 \leq t < \infty) \tag{1.1}$$

have found application in the mathematical modeling of electric pantograph dynamics [9, 18] and have even arisen in a partitioning problem in number theory [17]. In [11], Kato and MacLeod showed that this equation leads to a well-posed problem if $0 \leq \lambda \leq 1$ and examined properties of the solutions as $t \to \infty$. The dynamics and stability properties of solutions of a generalized form of (1.1) have been studied in [3, 7, 8, 10, 12, 21]. More recently, Brunner [4] has examined the attainable order of m-stage implicit (collocation based) Runge-Kutta methods for solving such equations. What is now clear is that properties that hold for constant delay type equations (i.e. which incorporate terms of the form $y(t-\sigma)$) do not necessarily carry over to even linear equations with proportional time delay and while constant time delays in reaction-diffusion systems have been studied extensively (see, e.g. [5, 6, 13, 14, 16, 19, 21, 23]), there appears to have been little investigation into the corresponding problems involving proportional delay.

The objective of this work is simply to employ some of the approaches adopted in [11] together with standard techniques for partial differential equations to determine whether the solution properties established for (1.1) will carry over to reaction-diffusion systems with general linear time delay

$$y_t(x,t) = \kappa \Delta y(x,t) + ay(x, \lambda t - \sigma), \qquad x \in \Omega, \;\; t > 0, \tag{1.2}$$

$$y(x,t) = 0, \qquad x \in \Gamma, \;\; t > 0, \tag{1.3}$$

$$y(x,s) = y^0(x,s), \qquad x \in \Omega, \;\; -\sigma \leq s \leq 0. \tag{1.4}$$

Here Ω is a nonempty bounded open set in $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$, Δ is the Laplacian operator, $\kappa > 0$ denotes the diffusivity constant, $\lambda > 0$, $\sigma \geq 0$ are parameters and $y^0(x,s)$ is an initial state in an appropriate function space. The extension of linear delays $\theta(t) = \lambda t - \sigma$ to more general expressions $\theta(t)$, introduces power delays of the form $\theta(t) = t^\lambda$ and so we also consider reaction-diffusion problems incorporating such terms

$$y_t(x,t) = \kappa \Delta y(x,t) + ay(x, t^\lambda), \qquad x \in \Omega, \;\; t > 0, \tag{1.5}$$

$$y(x,t) = 0, \qquad x \in \Gamma, \;\; t > 0, \tag{1.6}$$

$$y(x,0) = y^0(x), \qquad x \in \Omega. \tag{1.7}$$

(where the proportional term is in advance of time t for $0 < t < 1$).

Theorem 2.1 in Section 2 establishes the existence of unique solutions of (1.2)-(1.4) on $[0,T]$ for any T if $0 < \lambda \leq 1$ and for $0 < T < \sigma/(\lambda - 1)$ if $\lambda > 1$ and $\sigma > 0$. In Theorem 2.2 we show that if $0 < \lambda < 1$, then the solutions of (1.2)-(1.4) do not grow faster than a polynomial of degree $p = \frac{1}{\ln \lambda} \ln\left(\frac{\kappa\mu_0}{|a|}\right)$ and, moreover, if $|a| < \kappa\mu_0$, then $p < 0$ and the solutions decay to zero at a polynomial rate as $t \to \infty$. Theorems 3.1 and 3.2 in Section 3 establish the existence, uniqueness and polynomial growth of solutions of (1.5)-(1.7) for any

T if $0 < \lambda \leq 1$. A simple non-linear proportional time delay equation is considered in Section 4. The analysis of this simple example demonstrates that blowup behavior can be expected for such equations unless strong growth conditions are imposed on the nonlinear terms.

2 Evolution Equations with General Linear Delay

Let Ω be a bounded domain (nonempty, open, and connected set) in $\mathbb{R}^n$ with suitably smooth boundary $\Gamma = \partial\Omega$. We denote by $H^s(\Omega)$ the usual Sobolev space for fixed $s \in \mathbb{R}$ (see [1, 15]). For $s \geq 0$, $H_0^s(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in $H^s(\Omega)$, where $C_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions on Ω which have compact support in Ω. The norm of $L^2(\Omega)$ is denoted by $\|\cdot\|$. Let X be a Banach space and $a < b$. We denote by $C^n([a,b];X)$ the space of n times continuously differentiable functions defined on $[a,b]$ with values in X and endowed with the usual supremum norm. For brevity, we write $C^0([a,b];X)$ simply as $C([a,b];X)$.

Let A be the linear operator

$$Aw = \kappa\Delta w \tag{2.1}$$

with the domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. It is well known that A generates an analytic semigroup e^{At} on $L^2(\Omega)$. Problem (1.2)-(1.4) can be transformed into the following equivalent integral equation problem

$$y(t) = y^0(t) \qquad\qquad -\sigma \leq t \leq 0, \tag{2.2}$$

$$y(t) = e^{At}y^0 + a\int_0^t e^{A(t-s)}y(\lambda s - \sigma)ds \qquad\qquad t > 0, \tag{2.3}$$

where $y(t) = y(x,t)$, $y^0(t) = y^0(x,t)$. For any given $T > 0$ and $y^0 \in C([-\sigma,0];L^2(\Omega))$, the solution y of (2.2)-(2.3) with $y \in C([0,T];L^2(\Omega))$ is called a *mild* solution of problem (1.2)-(1.4). Using the standard bootstrap procedure for delay equations, it is straightforward to show that (1.2)-(1.4) is well-posed.

Theorem 2.1. *Let $\kappa > 0$, $\sigma \geq 0$ and a be real numbers.*

(i) If $0 < \lambda \leq 1$, then for any given $T > 0$ and initial condition $y^0(x,s) \in C([-\sigma,0];L^2(\Omega))$, problem (1.2)-(1.4) *has unique mild solution with*

$$y \in C([0,T];L^2(\Omega)). \tag{2.4}$$

Moreover, if $y^0(x,s) \in C^1([-\sigma,0];D(A))$, problem (1.2)-(1.4) *has unique classical solution with*

$$y \in C([0,T];H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega)). \tag{2.5}$$

(ii) If $\lambda > 1$ and $\sigma > 0$, The conclusions of (i) also hold for $0 < T < \sigma/(\lambda - 1)$.

Proof. Let $\{y_n\}$ $(n = 0,1,\cdots)$ be the sequence defined by

$$y_0(t) = \begin{cases} y^0(t) & \text{if } -\sigma \leq t \leq 0, \\ e^{At}y^0(0) + a\int_0^t e^{A(t-s)}y^0(\lambda s - \sigma)ds & \text{if } 0 \leq t \leq \sigma/\lambda, \\ e^{A(t-\sigma/\lambda)}y_0(\sigma/\lambda) & \text{if } t > \sigma/\lambda; \end{cases} \tag{2.6}$$

$$y_n(t) = \begin{cases} y^0(t) & \text{if } -\sigma \leq t \leq 0, \\ e^{At}y^0(0) + a\int_0^t e^{A(t-s)}y_{n-1}(\lambda s - \sigma)ds & \text{if } t \geq 0,\ n = 1,2,\cdots. \end{cases} \tag{2.7}$$

Then for $0 \le t \le \sigma/\lambda$

$$\begin{aligned} y_1(t) &= e^{At}y^0(0) + a\int_0^t e^{A(t-s)}y_0(\lambda s - \sigma)ds \\ &= e^{At}y^0(0) + a\int_0^t e^{A(t-s)}y^0(\lambda s - \sigma)ds \\ &= y_0(t), \end{aligned} \tag{2.8}$$

and

$$y_1(t) = y_0(t), \quad -\sigma \le t \le \sigma/\lambda. \tag{2.9}$$

In the same way, we deduce for $0 \le t \le \frac{\sigma}{\lambda} + \frac{\sigma}{\lambda^2}$ that

$$\begin{aligned} y_2(t) &= e^{At}y^0(0) + a\int_0^t e^{A(t-s)}y_1(\lambda s - \sigma)ds \\ &= e^{At}y^0(0) + a\int_0^t e^{A(t-s)}y_0(\lambda s - \sigma)ds \\ &= y_1(t), \end{aligned} \tag{2.10}$$

and

$$y_2(t) = y_1(t), \quad -\sigma \le t \le \frac{\sigma}{\lambda} + \frac{\sigma}{\lambda^2}. \tag{2.11}$$

Repeating the above procedure, gives

$$y_n(t) = y_{n-1}(t), \quad -\sigma \le t \le \frac{\sigma}{\lambda} + \frac{\sigma}{\lambda^2} + \cdots + \frac{\sigma}{\lambda^n}, \quad n = 1, 2, \cdots . \tag{2.12}$$

We now consider the cases: $\sigma > 0$ and $\sigma = 0$. Suppose $\sigma > 0$. Then, if $0 \le \lambda \le 1$ and $T > 0$ (or $\lambda > 1$ and $T < \sigma/(\lambda - 1)$), we can take n so large that

$$\sigma/\lambda + \sigma/\lambda^2 + \cdots + \sigma/\lambda^n > T$$

and

$$\begin{aligned} y_n(t) &= e^{At}y^0(0) + a\int_0^t e^{A(t-s)}y_{n-1}(\lambda s - \sigma)ds \\ &= e^{At}y^0(0) + a\int_0^t e^{A(t-s)}y_n(\lambda s - \sigma)ds. \end{aligned} \tag{2.13}$$

Thus y_n is a mild solution of problem (1.2)-(1.4). If $\sigma = 0$, then we have

$$\begin{aligned} \|y_1(t) - y_0(t)\| &= \left\| a\int_0^t e^{A(t-s)}y_0(\lambda s)ds \right\| \\ &= \left\| a\int_0^t e^{A(t-s+\lambda s)}y^0(0)ds \right\| \\ &\le |a|K\|y^0(0)\|t, \end{aligned} \tag{2.14}$$

where

$$K = \max_{0\le t\le \max\{T,\lambda T\}} \|e^{At}\|. \tag{2.15}$$

It follows that

$$\begin{aligned}
\|y_2(t)-y_1(t)\| &= \Big\|a\int_0^t e^{A(t-s)}(y_1(\lambda s)-y_0(\lambda s))ds\Big\| \\
&\le |a|K\int_0^t \|y_1(\lambda s)-y_0(\lambda s)\|ds \\
&\le |a|^2K^2\|y^0(0)\|\int_0^t \lambda s ds \\
&\le \frac{1}{2}|a|^2K^2\lambda\|y^0(0)\|t^2, \qquad (2.16)\\
\|y_n(t)-y_{n-1}(t)\| &\le \frac{1}{n!}|a|^nK^n\lambda^{1+2+\cdots+n-1}\|y^0(0)\|t^n, \quad n=1,2,\ldots, \qquad (2.17)
\end{aligned}$$

Hence, y_n converges to y in $C([0,T];L^2(\Omega))$ if $0<\lambda\le 1$. So, again problem (1.2)-(1.4) has mild solution y. Uniqueness is easily proved in the usual way and we omit the proof here.

To establish the regularity of the solutions, we set $z = y_t$. Then z satisfies (1.2)–(1.4) with a replaced by $a\lambda$ and the initial condition $y^0(x,s)$ replaced by $z^0(x,s) = y^0_s(x,s) \in C([-\sigma,0];L^2(\Omega))$. Thus, $z = y_t \in C([0,T];L^2(\Omega))$ and by the elliptic regularity, $y \in C([0,T];H^2(\Omega)\cap H^1_0(\Omega))\cap C^1([0,T];L^2(\Omega))$. □

We now utilize some of the techniques from [11] to show that the solutions of (1.2)-(1.4) do not grow faster than a polynomial. To this end, we let μ_0 be the smallest eigenvalue of $-\Delta$ with Dirichlet boundary condition

$$-\Delta\phi = \mu\phi, \qquad \text{in } \Omega, \tag{2.18}$$
$$\phi = 0, \qquad \text{on } \Gamma. \tag{2.19}$$

Theorem 2.2. *Suppose that $\kappa>0$, $\sigma\ge 0$, $0<\lambda<1$ and a is a real number. Then there exist positive constants $C = C(\kappa,a,\sigma,\lambda)$ and μ_0 such that the solution of (1.2)-(1.4) satisfies*

$$\|y(t)\| \le C\|y^0\|_{C([-\sigma,0];L^2(\Omega))}(t+1)^p, \quad \forall\, t\ge 0, \tag{2.20}$$
$$\|y_t(t)\| \le C\|y^0\|_{C^1([-\sigma,0];L^2(\Omega))}(t+1)^{p-1}, \quad \forall\, t\ge 0. \tag{2.21}$$

where

$$p = \frac{1}{\ln\lambda}\ln\Big(\frac{\kappa\mu_0}{|a|}\Big). \tag{2.22}$$

Furthermore, the exponent p given by (2.22) is optimal, that is, for any $\delta>0$, there exists a solution of (1.2)-(1.4) such that (2.20) with p replaced by $p-\delta$ does not hold.

Proof. The basic idea here is to transform the proportional delay $\lambda t-\sigma$ into the usual delay $s+\tau$ ($\tau<0$). Thus, we set

$$t = e^s - \frac{\sigma}{1-\lambda}, \qquad \tau = \ln\lambda < 0, \qquad u(x,s) = e^{-ps}y\Big(x, e^s - \frac{\sigma}{1-\lambda}\Big), \tag{2.23}$$

where p is given by (2.22). Since

$$\begin{aligned} y_t(x,t) &= [u_s(x,s)+pu(x,s)]\exp(ps+s), && (2.24)\\ y(x,\lambda t-\sigma) &= y\Big(x, \lambda e^s - \frac{\lambda\sigma}{1-\lambda} - \sigma\Big) \\ &= y\Big(x, e^{\tau+s} - \frac{\sigma}{1-\lambda}\Big) \\ &= e^{p(\tau+s)}u(x,\tau+s), && (2.25) \end{aligned}$$

it follows that u satisfies

$$\begin{aligned} & u_s(x,s) = \kappa e^s \Delta u(x,s) - pu(x,s) + ae^{p\tau}e^s u(x,\tau+s), \quad x\in\Omega, \ -\infty<s<\infty, && (2.26)\\ & u(0,s)=u(1,s)=0, \quad -\infty<s<\infty. && (2.27) \end{aligned}$$

Hence, to prove (2.20), it suffices to prove that there exists a positive constant $C = C(\kappa, a, \lambda)$ such that the solution u of (2.26)-(2.27) satisfies

$$\|u(s)\| \le C\|y^0\|_{C([-\sigma,0];L^2(\Omega))}, \quad \forall\, s \ge \ln\frac{\sigma}{1-\lambda}. \tag{2.28}$$

Let

$$\begin{aligned} s_0 &= \ln\frac{\sigma}{1-\lambda}, && (2.29)\\ M_n &= \max_{s_0-n\tau\le s\le s_0-n\tau-\tau} \|u(s)\|, \quad n=0,1,\ldots && (2.30) \end{aligned}$$

To prove (2.28), it suffices to establish that

$$M_n \le C(\kappa,a,\lambda)M_0, \qquad n=1,2,\ldots \tag{2.31}$$

and

$$M_0 \le C(\kappa,a,\lambda)\|y^0\|_{C([-\sigma,0];L^2(\Omega))}. \tag{2.32}$$

Integration by parts gives

$$\begin{aligned} \frac{d}{ds}\int_\Omega u^2(s)\,dx &= 2\int_\Omega u(s)u_s(s)\,dx \\ &= 2\kappa e^s\int_\Omega u(s)\Delta u(s)\,dx - 2p\int_\Omega u^2(s)\,dx - 2ae^{p\tau}e^s\int_\Omega u(s)u(s+\tau)\,dx \\ &= -2\kappa e^s\int_\Omega |\nabla u(s)|^2\,dx - 2p\int_\Omega u^2(s)\,dx - 2ae^{p\tau}e^s\int_\Omega u(s)u(s+\tau)\,dx \\ &\quad (\text{ note that } \mu_0\|u(x,t)\|^2 \le \|\nabla u(t)\|^2 \text{ for some } \mu_0>0) \\ &\le -2(p+\kappa\mu_0 e^s)\int_\Omega u^2(s)\,dx - 2ae^{p\tau}e^s\int_\Omega u(s)u(s+\tau)\,dx, && (2.33) \end{aligned}$$

which implies that

$$\frac{d}{ds}\Big(\exp(2ps+2\kappa\mu_0 e^s)\int_\Omega u^2(s)\,dx\Big) \le -2a\exp[p\tau+(2p+1)s+2\kappa\mu_0 e^s]\int_\Omega u(s)u(s+\tau)\,dx. \tag{2.34}$$

Integrating from $s_0-(n+1)\tau$ to s, we obtain

$$\begin{aligned}
&\exp(2ps+2\kappa\mu_0 e^s)\int_\Omega u^2(s)\,dx\\
&\le \exp(2p(s_0-(n+1)\tau)+2\kappa\mu_0\exp(s_0-(n+1)\tau))\int_\Omega u^2(s_0-(n+1)\tau)\,dx\\
&\quad -2a\int_{s_0-(n+1)\tau}^{s}\exp[p\tau+(2p+1)t+2\kappa\mu_0 e^t]\int_\Omega u(t)u(t+\tau)\,dxdt\\
&\le M_n^2\exp[2p(s_0-(n+1)\tau)+2\kappa\mu_0\exp(s_0-(n+1)\tau)]\\
&\quad +2|a|M_nM_{n+1}\int_{s_0-(n+1)\tau}^{s}\exp[p\tau+(2p+1)t+2\kappa\mu_0 e^t]dt. \qquad (2.35)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{s_0-(n+1)\tau}^{s}\exp[(2p+1)t+2\kappa\mu_0 e^t]dt\\
&=\frac{e^t}{2p+2\kappa\mu_0 e^t}\exp(2pt+2\kappa\mu_0 e^t)\Big|_{s_0-(n+1)\tau}^{s}-\int_{s_0-(n+1)\tau}^{s}\exp(2pt+2\kappa\mu_0 e^t)\frac{d}{dt}\Big(\frac{e^t}{2p+2\kappa\mu_0 e^t}\Big)dt\\
&=\frac{e^t}{2p+2\kappa\mu_0 e^t}\exp(2pt+2\kappa\mu_0 e^t)\Big|_{s_0-(n+1)\tau}^{s}-\int_{s_0-(n+1)\tau}^{s}\exp(2pt+2\kappa\mu_0 e^t)\frac{2pe^t}{(2p+2\kappa\mu_0 e^t)^2}dt\\
&\le\frac{e^t}{2p+2\kappa\mu_0 e^t}\exp(2pt+2\kappa\mu_0 e^t)\Big|_{s_0-(n+1)\tau}^{s}+2|p|\exp(2ps+2\kappa\mu_0 e^s)\int_{s_0-(n+1)\tau}^{s}\frac{e^t}{(2p+2\kappa\mu_0 e^t)^2}dt\\
&=\frac{e^t}{2p+2\kappa\mu_0 e^t}\exp(2pt+2\kappa\mu_0 e^t)\Big|_{s_0-(n+1)\tau}^{s}-\frac{|p|}{\kappa\mu_0}\exp(2ps+2\kappa\mu_0 e^s)\frac{1}{2p+2\kappa\mu_0 e^t}\Big|_{s_0-(n+1)\tau}^{s}\\
&=\Big(e^s-\frac{|p|}{\kappa\mu_0}\Big)\frac{\exp(2ps+2\kappa\mu_0 e^s)}{2p+2\kappa\mu_0 e^s}+\frac{|p|\exp(2ps+2\kappa\mu_0 e^s)}{\kappa\mu_0(2p+2\kappa\mu_0\exp(s_0-(n+1)\tau))}\\
&\quad-\frac{\exp(s_0-(n+1)\tau)}{2p+2\kappa\mu_0\exp(s_0-(n+1)\tau)}\exp[2p(s_0-(n+1)\tau)+2\kappa\mu_0\exp(s_0-(n+1)\tau)]\\
&\le\frac{e^s\exp(2ps+2\kappa\mu_0 e^s)}{2p+2\kappa\mu_0 e^s}+\frac{|p|\exp(2ps+2\kappa\mu_0 e^s)}{\kappa\mu_0(2p+2\kappa\mu_0\exp(s_0-(n+1)\tau))}\\
&\quad-\frac{\exp(s_0-(n+1)\tau)}{2p+2\kappa\mu_0\exp(s_0-(n+1)\tau)}\exp[2p(s_0-(n+1)\tau)+2\kappa\mu_0\exp(s_0-(n+1)\tau)]
\end{aligned}$$
$$(2.36)$$

and it follows from (2.35) that

$$
\begin{aligned}
&\exp(2ps + 2\kappa\mu_0 e^s) \int_\Omega u^2(s)\,dx \\
&\le M_n^2 \exp[2p(s_0 - (n+1)\tau) + 2\kappa\mu_0 \exp(s_0 - (n+1)\tau)] \\
&\quad - \frac{2|a| M_n M_{n+1} \exp(p\tau + s_0 - (n+1)\tau)}{2p + 2\kappa\mu_0 \exp(s_0 - (n+1)\tau)} \exp[2p(s_0 - (n+1)\tau) + 2\kappa\mu_0 \exp(s_0 - (n+1)\tau)] \\
&\quad + 2|a| e^{p\tau} M_n M_{n+1} \frac{e^s \exp(2ps + 2\kappa\mu_0 e^s)}{2p + 2\kappa\mu_0 e^s} \\
&\quad + 2|a| e^{p\tau} M_n M_{n+1} \frac{|p| \exp(2ps + 2\kappa\mu_0 e^s)}{\kappa\mu_0 (2p + 2\kappa\mu_0 \exp(s_0 - (n+1)\tau))}.
\end{aligned} \tag{2.37}
$$

Let

$$
f(s) = \frac{2|a| e^{p\tau} e^s}{2p + 2\kappa\mu_0 e^s}, \tag{2.38}
$$

$$
\sigma_n = 1 - f(s_0 - (n+1)\tau) + \frac{2|ap| e^{p\tau}}{\kappa\mu_0 (2p + 2\kappa\mu_0 \exp(s_0 - (n+1)\tau))}. \tag{2.39}
$$

We now establish that

$$
M_{n+1} \le (1 + \sigma_n) M_n. \tag{2.40}
$$

If $M_{n+1} \le M_n$, then (2.40) holds. Therefore, we may as well assume that $M_{n+1} > M_n$. Since the function

$$
f(s) = \frac{2|a| e^{p\tau} e^s}{2p + 2\kappa\mu_0 e^s}
$$

is increasing on $[0, \infty)$, it follows from (2.22) that

$$
f(s) \le \frac{|a| e^{p\tau}}{\kappa\mu_0} = 1. \tag{2.41}
$$

¿From (2.37) that

$$
\begin{aligned}
\int_\Omega u^2(s)\,dx \le\ & M_n M_{n+1} [1 - f(s_0 - (n+1)\tau)] \\
& \times \exp[2p(s_0 - (n+1)\tau) + 2\kappa\mu_0 \exp(s_0 - (n+1)\tau)] \exp(-2ps - 2\kappa\mu_0 e^s) \\
& + \frac{2|a| e^{p\tau} M_n M_{n+1} e^s}{2p + 2\kappa\mu_0 e^s} + \frac{2|ap| e^{p\tau} M_n M_{n+1}}{\kappa\mu_0 (2p + 2\kappa\mu_0 \exp(s_0 - (n+1)\tau))} \\
\le\ & M_n M_{n+1} [1 - f(s_0 - (n+1)\tau)] \\
& + \frac{|a| e^{p\tau} M_n M_{n+1}}{\kappa\mu_0} + \frac{|ap| e^{p\tau} M_n M_{n+1}}{\kappa\mu_0 (p + \kappa\mu_0 \exp(s_0 - (n+1)\tau))} \\
=\ & M_n M_{n+1} [1 - f(s_0 - (n+1)\tau)] \quad \text{(by (2.22))} \\
& + M_n M_{n+1} + \frac{|ap| e^{p\tau} M_n M_{n+1}}{\kappa\mu_0 (p + \kappa\mu_0 \exp(s_0 - (n+1)\tau))},
\end{aligned} \tag{2.42}
$$

which implies (2.40). Therefore, to prove (2.31), it suffices to prove that the product $\Pi_{n=1}^{\infty}(1+\sigma_n)$ is convergent or, equivalently, that the series $\sum_{n=1}^{\infty} \ln(1+\sigma_n)$ is convergent. This in turn is equivalent to showing that that the improper integral

$$\int_1^{\infty} \ln\Big(2 - \frac{|a|e^{p\tau}e^{-\tau s}}{p+\kappa\mu_0 e^{-\tau s}} + \frac{|ap|e^{p\tau}}{\kappa\mu_0(p+\kappa\mu_0 e^{-\tau s})}\Big) ds$$

is convergent which follows from

$$\begin{aligned}
&\lim_{s\to\infty} \frac{\ln\Big(2 - \frac{|a|e^{p\tau}e^{-\tau s}}{p+\kappa\mu_0 e^{-\tau s}} + \frac{|ap|e^{p\tau}}{\kappa\mu_0(p+\kappa\mu_0 e^{-\tau s})}\Big)}{(1+s^2)^{-1}} \\
&= \lim_{s\to\infty} \frac{\frac{d}{ds}\ln\Big(2 - \frac{|a|e^{p\tau}e^{-\tau s}}{p+\kappa\mu_0 e^{-\tau s}} + \frac{|ap|e^{p\tau}}{\kappa\mu_0(p+\kappa\mu_0 e^{-\tau s})}\Big)}{\frac{d}{dx}(1+s^2)^{-1}} \\
&= -\lim_{s\to\infty} \Big(\frac{p\tau|a|e^{p\tau}e^{-\tau s}}{(p+\kappa\mu_0 e^{-\tau s})^2} + \frac{\kappa\mu_0\tau|ap|e^{p\tau}e^{-\tau s}}{\kappa\mu_0(p+\kappa\mu_0 e^{-\tau s})^2}\Big)\frac{(1+s^2)^2}{2s} \\
&= 0.
\end{aligned} \tag{2.43}$$

It remains to prove (2.32). Since

$$\ln\frac{\sigma}{1-\lambda} \le s \le \ln\frac{\sigma}{1-\lambda} - \ln\lambda$$

is equivalent to

$$0 \le t = e^s - \frac{\sigma}{1-\lambda} \le \frac{\sigma}{\lambda},$$

it suffices to prove that

$$\max_{0\le t\le \sigma/\lambda} \|y(t)\| \le C(\kappa, a, \lambda)\|y^0\|_{C([-\sigma,0];L^2(\Omega))}. \tag{2.44}$$

Set

$$\varphi(t) = \max_{0\le s\le t} \|y(s)\|^2.$$

Let $C = C(\kappa, a, \lambda)$ denote a generic positive constant that may vary from line to line. Multiplying (1.2) by y and integrating over Ω by parts, we obtain for $0 \le t_1 \le t$

$$\begin{aligned}
\frac{d}{ds}(\|y(s)\|^2) &\le 2|a|\int_\Omega y(s)y(\lambda s - \sigma)\, dx \\
&\le C\|y^0\|_{C([-\sigma,0];L^2(\Omega))} + C\varphi(s).
\end{aligned} \tag{2.45}$$

Then

$$\begin{aligned}
\|y(t_1)\|^2 &\le (Ct_1+1)\|y^0\|_{C([-\sigma,0];L^2(\Omega))} + C\int_0^{t_1}\varphi(s)ds \\
&\le (Ct_1+1)\|y^0\|_{C([-\sigma,0];L^2(\Omega))} + C\int_0^{t}\varphi(s)ds.
\end{aligned} \tag{2.46}$$

and it follows that

$$\varphi(t) \leq (Ct+1)\|y^0\|_{C([-\sigma,0];L^2(\Omega))} + C\int_0^t \varphi(s)ds. \tag{2.47}$$

By Gronwall's inequality (see, e.g. [22, p.90]), we deduce that for $0 \leq t \leq \sigma/\lambda$

$$\varphi(t) \leq C(\kappa, a, \lambda)\|y^0\|_{C([-\sigma,0];L^2(\Omega))}. \tag{2.48}$$

To prove (2.21), we set $z = y_t$. Then z satisfies (1.2)–(1.4) with a replaced by $a\lambda$ and the initial condition $y^0(x,s)$ replaced by $z^0(x,s) = y^0_s(x,s) \in C([-\sigma,0];L^2(\Omega))$. Hence, we can apply (2.20) to z with p replaced by

$$p_1 = \frac{1}{\ln\lambda}\ln\left(\frac{\kappa\mu_0}{|a\lambda|}\right) = p-1 \tag{2.49}$$

and obtain (2.21).

To prove that p is optimal, we argue by contradiction. Suppose that there exists a $\delta_0 > 0$ such that

$$\|y(t)\| \leq C\|y^0\|_{C([-\sigma,0];L^2(\Omega))}(t+1)^{p-\delta_0}, \quad \forall\, t \geq 0 \tag{2.50}$$

for all solutions of (1.2)-(1.4). Let ϕ_0 be the eigenfunction of (2.18)-(2.19) corresponding to the eigenvalue μ_0 and $\psi(t)$ the solution of

$$\psi_t(t) = -\kappa\mu_0\psi(t) + a\psi(\lambda t), \tag{2.51}$$

$$\psi(0) = 1. \tag{2.52}$$

One can easily verify that $y = \phi_0\psi$ is a solution (1.2)-(1.4) with $\sigma = 0$ and $y^0 = \phi_0$. It then follows from (2.50) that

$$|\psi(t)| \leq C(t+1)^{p-\delta_0}, \quad \forall\, t \geq 0 \tag{2.53}$$

which, by Theorem 3 of [11], implies $\psi \equiv 0$. But this is impossible since $\psi(0) = 1$. □

Remark 2.1. If $|a| < \kappa\mu_0$, then $p < 0$ (note that $\ln\lambda < 0$). Therefore, the solution of (1.2)-(1.4) decays to zero at a polynomial rate as $t \to \infty$.

Remark 2.2. Let us compare the case $\lambda = 1$ with the case $0 < \lambda < 1$. For simplicity, we assume that $\sigma = 0$. If $\lambda = 1$ and $a > \kappa\mu_0$, problem (1.2)-(1.4) has an exponentially growing solution

$$y = \phi_0 \exp((a - \kappa\mu_0)t),$$

where ϕ_0 is the eigenfunction of (2.18)-(2.19) corresponding to μ_0. On the other hand, if $0 < \lambda < 1$, Theorem 2.2 shows that all solutions grow at most polynomially. This means that the proportionally delayed term $y(x, \lambda t)$ has a dissipative effect in the case $a > \kappa\mu_0$. For $a < \kappa\mu_0$, if $\lambda = 1$, all solutions of (1.2)-(1.4) satisfies

$$\|y(t)\| \leq \|y^0\| \exp((a - \kappa\mu_0)t).$$

while the solutions decay probably only polynomially if $0 < \lambda < 1$. This shows that the proportionally delayed term $y(x, \lambda t)$ has an anti-dissipative effect in the case $a < \kappa\mu_0$.

3 Evolution Equations with Power Time Delays

We now consider the linear equation with power time delay

$$y_t(x,t) = \kappa \Delta y(x,t) + a y(x,t^\lambda), \qquad x \in \Omega, \quad t > 0, \tag{3.1}$$
$$y(x,t) = 0, \qquad x \in \Gamma, \quad t > 0, \tag{3.2}$$
$$y(x,0) = y^0(x), \qquad x \in \Omega. \tag{3.3}$$

Note that the proportional term is in advance of time t for $0 < t < 1$. This problem could be transformed into a standard singular problem with the change of variable $\tau = t^\lambda$, but to conform with our approach to the general linear delay case, we transform it here into the following integral equation

$$y(t) = e^{At}y^0 + a\int_0^t e^{A(t-s)}y(s^\lambda)ds, \quad t > 0, \tag{3.4}$$

where A is the operator defined by (2.1). For any given $T > 0$ and $y^0 \in L^2(\Omega)$, the solution y of this integral equation with $y \in C([0,T]; L^2(\Omega))$ is called a *mild* solution of problem (3.1)-(3.3).

Theorem 3.1. *Suppose that $\kappa > 0$, $0 < \lambda \leq 1$ and a is a real number. Then for any given $T > 0$ and initial condition $y^0(x) \in L^2(\Omega)$, problem* (3.1)-(3.3) *has unique mild solution with*

$$y \in C([0,T]; L^2(\Omega)). \tag{3.5}$$

Moreover, if $y^0(x,s) \in D(A)$, then y is a classical solution with

$$y \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)). \tag{3.6}$$

Proof. We define a successive sequence $\{y_n\}$ $(n = 0, 1, \cdots)$ by

$$y_0(t) = e^{At}y^0, \tag{3.7}$$
$$y_n(t) = e^{At}y^0 + a\int_0^t e^{A(t-s)}y_{n-1}(s^\lambda)ds, \qquad n = 1, 2, \cdots. \tag{3.8}$$

We then have

$$\begin{aligned} \|y_1(t) - y_0(t)\| &= \left\| a\int_0^t e^{A(t-s)}y_0(s^\lambda)ds \right\| \\ &= \left\| a\int_0^t e^{A(t-s+s^\lambda)}y^0 ds \right\| \\ &\leq |a|K\|y^0\|t, \end{aligned} \tag{3.9}$$

where

$$K = \max_{0 \leq t \leq \max\{T, T^\lambda\}} \|e^{At}\|. \tag{3.10}$$

It therefore follows that

$$
\begin{aligned}
\|y_2(t)-y_1(t)\| &= \left\|a\int_0^t e^{A(t-s)}(y_1(s^\lambda)-y_0(s^\lambda))ds\right\| \\
&\le |a|K\int_0^t \|y_1(s^\lambda)-y_0(s^\lambda)\|ds \\
&\le |a|^2K^2\|y^0\|\int_0^t s^\lambda ds \\
&= \frac{1}{\lambda+1}|a|^2K^2\|y^0\|t^{\lambda+1}, \qquad (3.11)\\
\|y_n(t)-y_{n-1}(t)\| &\le \frac{|a|^nK^n\|y^0\|t^{1+\lambda+\cdots+\lambda^{n-1}}}{\prod_{i=1}^{n-1}(1+\lambda+\cdots+\lambda^i)}, \quad n=3,4,\ldots, \qquad (3.12)
\end{aligned}
$$

Hence, y_n converges to y in $C([0,T];L^2(\Omega))$ since $1+\lambda+\cdots+\lambda^{n-1}\le n$ for $0<\lambda\le 1$. So, problem (3.1)-(3.3) has unique mild solution y. The uniqueness can be easily proved in the usual way and we omit it.

To prove the regularity of solutions, we set $z=y_t$. Then z satisfies (3.1)-(3.3) with a replaced by $a\lambda t^{\lambda-1}$ and the initial condition $y^0(x)$ replaced by $z^0(x)=\kappa\Delta y^0(x)+ay^0(x)\in L^2(\Omega)$. We can readily check that the above arguments are also true if a is replaced by $a\lambda t^{\lambda-1}$. Therefore, we have $z=y_t\in C([0,T];L^2(\Omega))$ and then, by the elliptic regularity, we deduce that $y\in C([0,T];H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$. □

Using an approach similar to that in Theorem 2.2, we derive the following result regarding the asymptotic behavior of solutions of (3.1)-(3.3).

Theorem 3.2. *Suppose that $\kappa>0,\ 0<\lambda<1$ and a is a real number. Let*

$$
p>\frac{1}{\lambda(1-\lambda)}. \qquad (3.13)
$$

Then there exists a positive constant $C=C(\kappa,a,\lambda)$ such that the solution of (3.1)-(3.3) satisfies

$$
\|y(t)\| \ \le\ C\|y^0\|(t+1)^p, \quad \forall\, t\ge 0 \qquad (3.14)
$$

Proof. The proof is similar to that of Theorem 2.2. Set

$$
t=e^{e^s}, \qquad \tau=\ln\lambda<0, \qquad u(x,s)=e^{-pe^s}y\Big(x,e^{e^s}\Big). \qquad (3.15)
$$

Since

$$
\begin{aligned}
y_t(x,t) &= (u_s(x,s)+pe^su(x,s))\exp((p-1)e^s-s), \qquad (3.16)\\
y(x,t^\lambda) &= y\Big(x,e^{\lambda e^s}\Big)\\
&= e^{p\lambda e^s}u(x,\tau+s), \qquad (3.17)
\end{aligned}
$$

it follows that u satisfies

$$u_s(x,s) = \kappa \exp(s+e^s)\Delta u(x,s) - pe^s u(x,s) + a\exp[s+e^s+(\lambda-1)pe^s]u(x,\tau+s) \quad (3.18)$$
$$x \in \Omega, \quad -\infty < s < \infty,$$
$$u(0,s) = u(1,s) = 0 \qquad -\infty < s < \infty. \quad (3.19)$$

As in the proof of Theorem 2.2, to prove (3.14), it suffices to prove that the solution u of (3.18)-(3.19) is bounded. Set

$$M_n = \max_{-n\tau \le s \le -n\tau-\tau} \|u(s)\|, \quad n = 0, 1, \dots \quad (3.20)$$

It suffices to prove M_n is bounded. Integrating by parts, we obtain

$$\begin{aligned}
\frac{d}{ds}\int_\Omega u^2(s)\,dx &= 2\int_\Omega u(s)u_s(s)\,dx \\
&= 2\kappa\exp(s+e^s)\int_\Omega u(s)\Delta u(s)\,dx - 2pe^s\int_\Omega u^2(s)\,dx \\
&\quad - 2a\exp[s+e^s+(\lambda-1)pe^s]\int_\Omega u(s)u(s+\tau)\,dx \\
&= -2\kappa\exp(s+e^s)\int_\Omega |\nabla u(s)|^2\,dx - 2pe^s\int_\Omega u^2(s)\,dx \\
&\quad - 2a\exp[s+e^s+(\lambda-1)pe^s]\int_\Omega u(s)u(s+\tau)\,dx \\
&\text{(again with } \mu_0\|u(x,t)\|^2 \le \|\nabla u(t)\|^2 \text{)} \\
&\le -2[pe^s + \kappa\mu_0\exp(s+e^s)]\int_\Omega |u(s)|^2\,dx \\
&\quad - 2a\exp[s+e^s+(\lambda-1)pe^s]\int_\Omega u(s)u(s+\tau)\,dx, \qquad (3.21)
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{d}{ds}\Big(\exp(2pe^s + 2\kappa\mu_0\exp(e^s))\int_\Omega u^2(s)\,dx\Big) \\
&\le -2a\exp[2pe^s + 2\kappa\mu_0\exp(e^s) + s + e^s + (\lambda-1)pe^s]\int_\Omega u(s)u(s+\tau)\,dx. \qquad (3.22)
\end{aligned}$$

Integrating (3.22) from $-(n+1)\tau$ to s, we obtain

$$\begin{aligned}
&\exp(2pe^t + 2\kappa\mu_0\exp(e^t))\int_\Omega u^2(t)\,dx\Big|_{-(n+1)\tau}^{s} \\
&\le -2a\int_{-(n+1)\tau}^{s} \exp[2pe^t + 2\kappa\mu_0\exp(e^t) + t + e^t + (\lambda-1)pe^t]\int_\Omega u(t)u(t+\tau)\,dxdt \\
&\le -2\tau|a|M_nM_{n+1}\exp[2pe^s + 2\kappa\mu_0\exp(e^s) + s + e^s + (\lambda-1)pe^{-(n+1)\tau}], \qquad (3.23)
\end{aligned}$$

which implies that

$$M_{n+1}^2 \le M_n^2 - 2\tau|a|M_nM_{n+1}\exp[-(n+2)\tau + e^{-(n+2)\tau} + (\lambda-1)pe^{-(n+1)\tau}]. \quad (3.24)$$

Set

$$\delta = e^{-\tau} + (\lambda - 1)p < 0 \qquad \text{(by (3.13))}. \tag{3.25}$$

We then have

$$M_{n+1} \le [1 + 2|\tau||a|e^{-\tau} \exp[-(n+1)\tau + \delta e^{-(n+1)\tau}]]M_n. \tag{3.26}$$

If $M_{n+1} \le M_n$, then (3.26) clearly holds. If $M_{n+1} > M_n$, then (3.26) follows from (3.24). Therefore, to prove that M_n is bounded, it suffices to prove that the product

$$\Pi_{n=1}^{\infty}[1 + 2|\tau||a|e^{-\tau} \exp[-(n+1)\tau + \delta e^{-(n+1)\tau}]]$$

is convergent, or equivalently, to show that the improper integral

$$\int_1^{\infty} \ln\Big(1 + 2|\tau||a|e^{-\tau} \exp[-\tau s + \delta e^{-\tau s}]\Big) ds$$

is convergent. But this follows as in the proof of Theorem 2.2. □

4 Nonlinear Equations

We now consider the nonlinear proportional delay boundary value problem

$$y_t(x,t) = \kappa y_{xx}(x,t) + ay(x,t)[1 - y(x,\lambda t)], \qquad 0 < x < 1, \;\; t > 0, \tag{4.1}$$

$$y(0,t) = y(1,t) = 0, \qquad t > 0, \tag{4.2}$$

$$y(x,0) = y^0(x), \qquad 0 < x < 1. \tag{4.3}$$

Again, the problem can be transformed into the following integral equation

$$y(t) = e^{At}y^0 + a\int_0^t e^{A(t-s)}y(s)[1 - y(\lambda s)]ds, \quad t > 0, \tag{4.4}$$

where A is the operator defined by (2.1). The existence and uniqueness of a local solution of (4.1)-(4.3) can be readily established using the Banach contraction fixed point theorem, but a determination of the of asymptotic behavior of such solutions presents some serious difficulties. Indeed, (4.1-4.2) it may not have a global solution as the following simple example demonstrates.

Let

$$y^0(x) = 10\sin(5\pi x).$$

For $a = 5$, it can be seen from Figure 4.1 that the energy

$$E(t) = \int_0^1 y^2(x,t)dx,$$

of the solution without delay (i.e. $\lambda = 1$) decays rapidly to zero while the energy of the solution with delay (i.e. $\lambda = 0.5$) decreases initially and then increases rapidly after $t = 2.5$ and may blow up within finite time.

The proofs of Theorems 2.1, 2.2, 3.1 and 3.2 provide the foundation for applying standard bootstrap methods (cf [20]) to questions of existence, uniqueness and properties of solutions of semilinear (and more general) proportional delay equations of the form

$$y_t(x,t) = \kappa\Delta y(x,t) + f(t, y(x,t), y(x,\theta(t))), \qquad x \in \Omega, \qquad t > 0.$$

and this will be the focus of a future work.

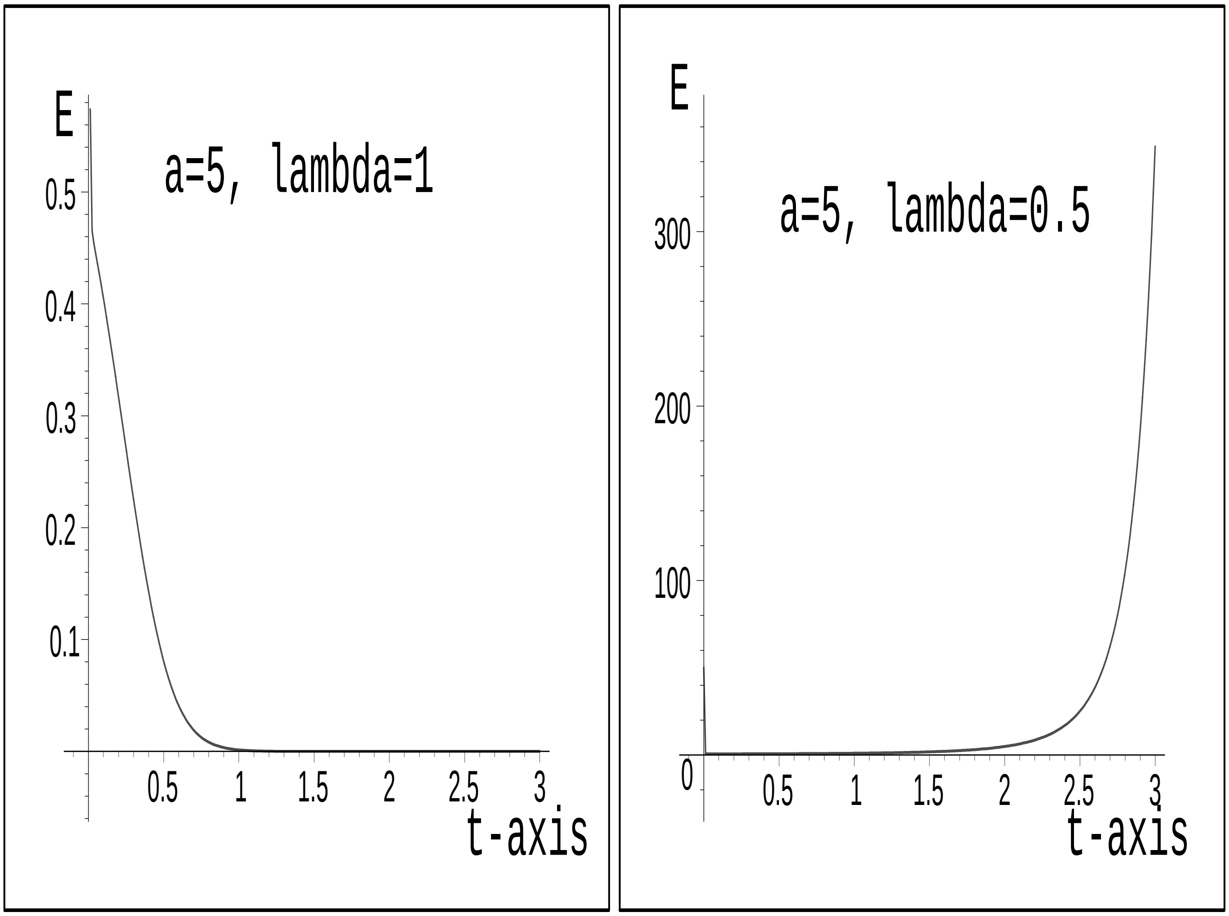

Figure 4.1: Energy curves of solutions of the equation $y_t(x,t) = y_{xx}(x,t) + ay(x,t)[1 - y(x,\lambda t)]$ with the initial condition $y^0(x) = 10\sin(5\pi x)$.

References

[1] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] W.F. Ames, *Numerical Methods for Partial Differential Equations,* Second edition, Academic Press, New York, 1977.

[3] H. Brunner, The numerical solution of neutral Volterra integro-differential equations with delay arguments, *Ann. Numer. Math.,* **1** (1994) 309-322.

[4] H. Brunner, On the discretization of differential and Volterra integral equations with variable delay, *BIT,* **37;1** (1997) 1-12.

[5] G. Friesecke, Convergence to equilibrium for delay-diffusion equations with small delay, *J. Dynamics and Differential Equations* **5** (1993), 89-103.

[6] G. Friesecke, Exponentially growing solutions for a delay-diffusion equation with negative feedback, *J. Differential Equations* **98** (1992), no. 1, 1–18.

[7] A. Iserles, Y. Liu, On neutral functional-differential equations with proportional delays, *J. Math. Anal. Appl.* **207** (1997), no. 1, 73–95.

[8] A. Iserles, On nonlinear delay differential equations, *Trans. Amer. Math. Soc.* **344** (1994), no. 1, 441–477.

[9] A. Iserles, On the generalized pantograph functional differential equation, *Europ. J. Appl. Math.* **4** (1993), 1-38.

[10] E. Ishiwata, On the attainable order of collocation methods for the neutral functional-differential equations with proportional delays, *Computing* **64** (2000), no. 3, 207–222.

[11] T. Kato and J.B. McLeod, The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$, *Bull. Amer. Math. Soc.* **77** (1971), 891-937.

[12] R. Kress, *Numerical Analysis*, Springer, New York, 1998.

[13] I. Lasiecka, Unified theory for abstract parabolic boundary problems– a semigroups approach, *Appl. Math. Optim.* **6** (1980), 287-333.

[14] X. J. Li and K. S. Liu, The effect of small time delays in the feedbacks on boundary stabilization, *Science in China* **36** (1993), 1435-1443.

[15] J. L. Lions and E. Magenes, *Non-homogeneous Boundary value Problems and Applications*, Vol.1, Springer-Verlag, Berlin, 1972.

[16] S. Luckhaus, Global boundedness for a delay differential equation, *Trans. Amer. Math. Soc.* **294** (1986), 767-774.

[17] K. Mahler, On a special functional equation, *J. London Math. Soc.* **15** (1940), 115-123.

[18] J. R. Ockendon and A. B. Tayler, The dynamics of a current collection system for an electric locomotive, *Proc. Roy. Soc. London Ser. A* **322** (1971), 447-468.

[19] S. M. Oliva, Reaction-diffusion equations with nonlinear boundary delay, *J. Dynamics and Differential Equations* **11** (1999), 279-296.

[20] S. Ruan and John C. Clements, Existence and uniqueness of solutions of retarded quasi-linear wave equations, *Fields Institute Communications* **25** (2000), 473-483.

[21] N. Takama, Y. Muroya and E. Ishiwata, On the attainable order of collocation methods for delay differential equations with proportional delay. *BIT* **40** (2000), no. 2, 374–394.

[22] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, 2nd ed., Springer-Verlag, New York, 1997.

[23] J. Wu, *Theory and Applications of Partial Functional Differential Equations,* Springer-Verlag, New York, 1996.