Published in *Quarterly of Applied Math.* 58 (2000), 37-68.

Exact Controllability for Problems of Transmission of the Plate Equation with Lower-order Terms

Weijiu Liu and Graham H. Williams

Department of Mathematics, University of Wollongong
Northfields Avenue, Wollongong
NSW 2522, Australia
E-mail: w.liu@uow.edu.au; ghw@uow.edu.au

Abstract. We consider the exact controllability for the problem of transmission of the plate equation with lower-order terms. Using Lions' Hilbert Uniqueness Method (HUM for short), we show that the system is exactly controllable in $L^2(\Omega) \times H^{-2}(\Omega)$. We also obtain some uniqueness theorems for the problem of transmission of the plate equation and for the operator $a(x)\Delta^2 + q$.

Key Words: Exact controllability, Plate equation, Problem of transmission.

AMS subject classification: 93B05, 35B37.

§0. Introduction

This paper aims to consider the exact controllability for the problem of transmission of the plate equation with lower-order terms

$$\begin{cases} y'' + a(x)\Delta^2 y + qy = 0 & \text{in } Q, \\ y(0) = y^0, \quad y'(0) = y^1 & \text{in } \Omega, \\ y_2 = 0, \qquad \dfrac{\partial y_2}{\partial \nu} = \phi & \text{on } \Sigma, \\ y_1 = y_2, \quad \dfrac{\partial y_1}{\partial \nu} = \dfrac{\partial y_2}{\partial \nu} & \text{on } \Sigma_1, \\ a_1 \Delta y_1 = a_2 \Delta y_2, \quad a_1 \dfrac{\partial \Delta y_1}{\partial \nu} = a_2 \dfrac{\partial \Delta y_2}{\partial \nu} & \text{on } \Sigma_1. \end{cases} \tag{0.1}$$

In (0.1), Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with suitably smooth boundary $\Gamma = \partial\Omega$, and Ω_1 a bounded domain with $\overline{\Omega}_1 \subset \Omega$ and suitably smooth boundary $\Gamma_1 = \partial\Omega_1$; $\Omega_2 = \Omega - \Omega_1$, $Q = \Omega \times (0,T)$, $Q_1 = \Omega_1 \times (0,T)$, $Q_2 = \Omega_2 \times (0,T)$, $\Sigma = \Gamma \times (0,T)$, $\Sigma_1 = \Gamma_1 \times (0,T)$ for $T > 0$; ν is the unit normal of Γ or Γ_1 pointing towards the exterior of Ω or Ω_1; $y' = \dfrac{\partial y}{\partial t}$, $y(0) = y(x,0)$, $y'(0) = y'(x,0)$, $y_1 = y|_{\Omega_1}$, $y_2 = y|_{\Omega_2}$; $q(x,t)$ is a given function on Q satisfying

$$q \in L^\infty(Q), \tag{0.2}$$

and $a(x)$ is given by

$$a(x) = \begin{cases} a_1, & x \in \Omega_1, \\ a_2, & x \in \Omega_2, \end{cases} \tag{0.3}$$

where a_1 and a_2 are positive constants.

To be precise, the problem is: *For suitable $T > 0$ and every initial (y^0, y^1) (given in a suitable Hilbert space), we want to find a corresponding control ϕ driving system (0.1) to rest at time T, that is, such that the solution $y(x,t;\phi)$ of (0.1) satisfies*

$$y(x,T;\phi) = y'(x,T;\phi) = 0 \qquad \text{in } \Omega. \tag{0.4}$$

Since the problem is linear, this is equivalent to steering the system to any state. This question was raised by Lions (see [10], p.395) in the case of $q \equiv 0$.

There has been extensive work (see [1], [4]-[11]) over the past ten years on the problem of exact controllability for the plate equation. However, Concerning the problem of transmission, there has been no study so far. In this paper, using Lions' Hilbert Uniqueness Method (HUM for short), we will show that under certain assumptions made on Ω_1, $a(x)$, and $q(x)$ we can find a control $\phi \in L^2(\Sigma)$ steering system (0.1) from any initial state $(y^0, y^1) \in L^2(\Omega) \times H^{-2}(\Omega)$ to rest.

Meanwhile, we also obtain some uniqueness theorems for the problem of transmission of the plate equation and for the operator $a(x)\Delta^2 + q$, which themself are of significant importance. These theorems are in part the answer to the open question raised by Zuazua in [14].

Throughout this paper, in addition to the above notation about Ω and Ω_1, we also adopt the following notation. Let $x^0 \in \mathbb{R}^n$ and set

$$\begin{gathered} m(x) = x - x^0 = (x_k - x_k^0), \\ \Gamma(x^0) = \{x \in \Gamma \ : \ m(x) \cdot \nu(x) = m_k(x)\nu_k(x) > 0\}, \\ \Gamma_*(x^0) = \Gamma - \Gamma(x^0) = \{x \in \Gamma \ : \ m(x) \cdot \nu(x) \le 0\}, \\ \Sigma(x^0) = \Gamma(x^0) \times (0,T), \\ \Sigma_*(x^0) = \Gamma_*(x^0) \times (0,T), \\ R(x^0) = \max_{x \in \bar{\Omega}} \mid m(x) \mid = \max_{x \in \bar{\Omega}} \mid \sum_{k=1}^{n} (x_k - x_k^0)^2 \mid^{\frac{1}{2}} . \end{gathered}$$

$H^s(\Omega)$ always denotes the usual Sobolev space and $\| \cdot \|_{s,\Omega}$ denotes its norm for any $s \in \mathbb{R}$. Let X be a Banach space. We denote by $C^k([0,T],X)$ the space of all k times continuously differentiable functions defined on $[0,T]$ with values in X, and write $C([0,T],X)$ for $C^0([0,T],X)$.

The plan for this paper is as follows. In section 1, using the variational methods, we briefly discuss the well-posedness for the problem of transmission. Section 2 is devoted to the discussion of the boundary regularity of solutions . Section 3 concerns nonhomogeneous boundary value problems. In section 4, the key estimates for the solutions (i.e., the so-called " observability inequality") are obtained. Finally, in section 5, the theorems of exact controllability are given.

§1. Well-posedness

We shall use the variational methods to solve the problem

$$\begin{cases} u'' + a(x)\Delta^2 u + q(x,t)u = f & \text{in } Q, \\ u(0) = u^0, \quad u'(0) = u^1 & \text{in } \Omega, \\ u_2 = \dfrac{\partial u_2}{\partial \nu} = 0, & \text{on } \Sigma, \\ u_1 = u_2, \quad \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} & \text{on } \Sigma_1, \\ a_1 \Delta u_1 = a_2 \Delta u_2, \quad a_1 \dfrac{\partial \Delta u_1}{\partial \nu} = a_2 \dfrac{\partial \Delta u_2}{\partial \nu} & \text{on } \Sigma_1, \end{cases} \tag{1.1}$$

where $u_1 = u|_{\Omega_1}$ and $u_2 = u|_{\Omega_2}$.

Set

$$H^4(\Omega,\Gamma_1) = \Big\{u \ : \ u \in H_0^2(\Omega);\ u_i \in H^4(\Omega_i),\ i=1,2; \qquad a_1\Delta u_1 = a_2\Delta u_2 \text{ and } a_1\frac{\partial \Delta u_1}{\partial \nu} = a_2\frac{\partial \Delta u_2}{\partial \nu} \text{ on } \Gamma_1\Big\} \tag{1.2}$$

with the norm

$$\| u \|_{H^4(\Omega,\Gamma_1)} = \Big(\| u_1 \|_{4,\Omega_1}^2 + \| u_2 \|_{4,\Omega_2}^2\Big)^{\frac{1}{2}}. \tag{1.3}$$

It is well known that the norms $\| u \|_{4,\Omega}$ and $\| \Delta^2 u \|_{0,\Omega}$ on $H^4(\Omega)\cap H_0^2(\Omega)$ are equivalent since Δ^2 is an isomorphism from $H^4(\Omega)\cap H_0^2(\Omega)$ into $L^2(\Omega)$ (see [12], Vol I, Chapter 2, p.165). We generalize this result to the transmission case $H^4(\Omega,\Gamma_1)$. We first consider the regularity of the solution of

$$\begin{cases} a(x)\Delta^2 u = f & \text{in } \Omega,\\ u_2 = \dfrac{\partial u_2}{\partial \nu} = 0 & \text{on } \Gamma,\\ u_1 = u_2, \quad \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} & \text{on } \Gamma_1,\\ a_1\Delta u_1 = a_2\Delta u_2, \quad a_1\dfrac{\partial \Delta u_1}{\partial \nu} = a_2\dfrac{\partial \Delta u_2}{\partial \nu} & \text{on } \Gamma_1. \end{cases} \tag{1.4}$$

Lemma 1.1. *The solutions of (1.4) belongs to $H^4(\Omega,\Gamma_1)$ for $f\in L^2(\Omega)$.*

Proof. The solution u of (1.4) can be written as

$$u = \begin{cases} u_1, & x\in\Omega_1,\\ u_2, & x\in\Omega_2, \end{cases}$$

where $u_1,\ u_2$ are respectively the solutions of

$$\begin{cases} a_2\Delta^2 u_2 = f & \text{in } \Omega,\\ u_2 = \dfrac{\partial u_2}{\partial \nu} = 0 & \text{on } \Gamma, \end{cases}$$

and

$$\begin{cases} a_1\Delta^2 u_1 = f & \text{in } \Omega_1,\\ u_1 = u_2, \quad \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} & \text{on } \Gamma_1,\\ a_1\Delta u_1 = a_2\Delta u_2, \quad a_1\dfrac{\partial \Delta u_1}{\partial \nu} = a_2\dfrac{\partial \Delta u_2}{\partial \nu} & \text{on } \Gamma_1. \end{cases}$$

It follows from the elliptic regularity that $u_2\in H^4(\Omega)$. By the trace theorem we deduce $u_2 \in H^{\frac{7}{2}}(\Gamma_1)$ and $\dfrac{\partial u_2}{\partial \nu} \in H^{\frac{5}{2}}(\Gamma_1)$. Again by the elliptic regularity we have $u_1\in H^4(\Omega_1)$. Thus $u\in H^4(\Omega,\Gamma_1)$. □

Lemma 1.2. *Suppose the boundaries Γ and Γ_1 of Ω and Ω_1 are of class C^4. Then the norm $\| u \|_{H^4(\Omega,\Gamma_1)}$ on $H^4(\Omega,\Gamma_1)$ defined by (1.3) is equivalent to*

$$| u |_{H^4(\Omega,\Gamma_1)} = \Big(\| \Delta^2 u_1 \|_{0,\Omega_1}^2 + \| \Delta^2 u_2 \|_{0,\Omega_2}^2\Big)^{\frac{1}{2}}. \tag{1.5}$$

Proof. The sesquilinear form $\alpha(u,v)$ associated with problem (1.4) is

$$\alpha(u,v) = \int_\Omega a(x)\Delta u \Delta v \ dx, \qquad \forall u,v\in H_0^2(\Omega).$$

It is easy to see that the sesquilinear form α is coercive on $H_0^2(\Omega)$. Let A be the operator associated with the sesquilinear form α by

$$\alpha(u,v) = \langle Au, v\rangle, \qquad \forall u, v \in H_0^2(\Omega),$$

where $\langle \cdot,\cdot\rangle$ denotes the dual product between the spaces $H_0^2(\Omega)$ and $H^{-2}(\Omega)$.
By the Lax-Milgram theorem, A is an isomorphism of $H_0^2(\Omega)$ onto $H^{-2}(\Omega)$. Set

$$D(A) = \{u \in H_0^2(\Omega) \ : \ Au \in L^2(\Omega)\},$$

then by Lemma 1.1 we have

$$D(A) = H^4(\Omega, \Gamma_1).$$

By Proposition 9 of [2, p.370], $H^4(\Omega,\Gamma_1)$ provided with the norm of the graph

$$\| u \|_{D(A)} = \left(\| u \|_{L^2(\Omega)}^2 + \| Au \|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}$$

is a Banach space. In addition, $H^4(\Omega,\Gamma_1)$ with the norm $\| \cdot \|_{H^4(\Omega,\Gamma_1)}$ is also a Banach space, and the norm $\| \cdot \|_{H^4(\Omega,\Gamma_1)}$ is stronger than $\| \cdot \|_{D(A)}$. By the Banach open mapping theorem, these two norms $\| u \|_{H^4(\Omega,\Gamma_1)}$ and $\|u\|_{D(A)}$ on $H^4(\Omega,\Gamma_1)$ are equivalent. Again by the Banach open mapping theorem, A is an isomorphism of $H^4(\Omega,\Gamma_1)$ onto $L^2(\Omega)$ with $H^4(\Omega,\Gamma_1)$ provided with the norm of graph since $A : H^4(\Omega,\Gamma_1) \to L^2(\Omega)$ is continuous, one-to-one, and onto. Consequently, A is also an isomorphism of $H^4(\Omega,\Gamma_1)$ onto $L^2(\Omega)$ with $H^4(\Omega,\Gamma_1)$ provided with the norm $\| \cdot \|_{H^4(\Omega,\Gamma_1)}$. It therefore follows that the norm $\| \ u \|_{H^4(\Omega,\Gamma_1)}$ is equivalent to $| \ u \ |_{H^4(\Omega,\Gamma_1)}$ on $H^4(\Omega,\Gamma_1)$ since $A = a(x)\Delta^2$ on $H^4(\Omega,\Gamma_1)$. □

Lemma 1.3. *$H^4(\Omega,\Gamma_1)$ is dense in $H_0^2(\Omega)$.*

Proof. Let $H_0^2(\Omega)$ be provided with the following scalar product

$$\langle u, v\rangle = \int_\Omega a(x)\Delta u \Delta v dx, \qquad \forall u, v \in H_0^2(\Omega),$$

which is equivalent to the usual one.

Let $w \in H_0^2(\Omega)$ be such that

$$\langle u, w\rangle = 0, \qquad \forall u \in H^4(\Omega,\Gamma_1),$$

then

$$0 = \int_\Omega a(x)\Delta u \Delta w dx = \int_\Omega a(x) w \Delta^2 u dx.$$

By Lemma 1.1, $a(x)\Delta^2 u$ runs over all $L^2(\Omega)$ when u runs over all $H^4(\Omega,\Gamma_1)$. Thus $w = 0$. It therefore follows from Hahn-Banach theorem that $H^4(\Omega,\Gamma_1)$ is dense in $H_0^2(\Omega)$. □

Lemma 1.3 is not obvious since $C_0^\infty(\Omega) \not\subset H^4(\Omega,\Gamma_1)$.

Let λ_1 be the smallest eigenvalues of the operator Δ^2 with Dirichlet homogeneous boundary conditions on $L^2(\Omega)$, that is,

$$\begin{cases} \Delta^2 u = \lambda_1 u & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma. \end{cases} \tag{1.6}$$

Then

$$\|u\|_0 \le \frac{1}{\sqrt{\lambda_1}} \|\Delta u\|_0, \qquad \forall u \in H_0^2(\Omega). \tag{1.7}$$

Since we have for $u \in H^2(\Omega) \cap H_0^1(\Omega)$

$$\int_\Omega |\nabla u|^2 dx = \int_\Omega -u\Delta u dx \leq \|u\|_0 \|\Delta u\|_0, \tag{1.8}$$

we deduce from (1.7) that

$$\|\nabla u\|_0 \leq \frac{1}{\lambda_1^{1/4}} \|\Delta u\|_0, \qquad \forall u \in H_0^2(\Omega). \tag{1.9}$$

We define the energy of system (1.1) by

$$E(u,t) = \frac{1}{2} \int_\Omega [| u'(x,t) |^2 + a(x) | \Delta u |^2] dx. \tag{1.10}$$

Theorem 1.4. *(i) Suppose the boundaries Γ and Γ_1 are of class C^2. Then for every initial condition $(u^0, u^1) \in H_0^2(\Omega) \times L^2(\Omega)$ and $f \in L^1(0,T; L^2(\Omega))$, problem (1.1) has a unique weak solution with*

$$u(t) \in C([0,T]; H_0^2(\Omega)) \cap C^1([0,T]; L^2(\Omega)). \tag{1.11}$$

Moreover, there exists a constant $c > 0$ such that for all $t \in [0,T]$

$$\| u(t) \|_{2,\Omega} + \| u'(t) \|_{0,\Omega} \leq c\Big[\| u^0 \|_{2,\Omega} + \| u^1 \|_{0,\Omega} + \| f \|_{L^1(0,T;L^2(\Omega))}\Big]. \tag{1.12}$$

(ii) Suppose the boundary Γ of Ω and the boundary Γ_1 of Ω_1 are of class C^4. Assume that $q \in L^\infty(0,T; W^{2,\infty}(\Omega))$. Then for any initial condition $(u^0, u^1) \in H^4(\Omega, \Gamma_1) \times H_0^2(\Omega)$ and $f \in L^1(0,T; H_0^2(\Omega))$, the problem (1.1) has a unique strong solution with

$$u(t) \in C([0,T]; H^4(\Omega, \Gamma_1)) \cap C^1([0,T]; H_0^2(\Omega)). \tag{1.13}$$

Moreover, there exists a constant $c > 0$ such that for all $t \in [0,T]$

$$\| u'(t) \|_{2,\Omega} + \|u(t)\|_{H^4(\Omega,\Gamma_1)} \tag{1.14}$$

$$\leq c\Big[\| u^1 \|_{2,\Omega} + \|u^0\|_{H^4(\Omega,\Gamma_1)} + \| f \|_{L^1(0,T;H_0^2(\Omega))}\Big].$$

Theorem 1.4 can be proved by the variational methods (see [3], Chapter XVIII) and Lemma 1.2.

§2. Boundary Regularity

In this section we discuss the boundary regularity of the solution of (1.1). To this end, we first establish the following important identity that plays the key role in obtaining the boundary estimations for the solutions. These estiamtions are the essential part in the applications of HUM.

Lemma 2.1. *Suppose the boundary Γ of Ω is of class C^3 and the boundary Γ_1 of Ω_1 is of class C^4. Let $\rho = (\rho_k)$ be a vector field in $[C^2(\overline{\Omega})]^n$. Suppose u is the strong solution of (1.1).*

Then the following identity holds:

$$
\begin{aligned}
&\frac{1}{2}\int_{\Sigma} a_2\rho_k\nu_k \mid \Delta u_2 \mid^2 \, d\Sigma \\
&= \Big(u'(t), \rho_k \frac{\partial u(t)}{\partial x_k}\Big)\Big|_0^T + \frac{1}{2}\int_Q \frac{\partial \rho_k}{\partial x_k}[\mid u' \mid^2 - a(x) \mid \Delta u \mid^2] dxdt \\
&\quad + \int_Q a(x)\Delta\rho_k \Delta u \frac{\partial u}{\partial x_k} dxdt + 2\int_Q a(x)\frac{\partial \rho_k}{\partial x_j}\Delta u \frac{\partial^2 u}{\partial x_k \partial x_j} dxdt \\
&\quad - \int_Q f\rho_k \frac{\partial u}{\partial x_k} dxdt + \int_Q q\rho_k u \frac{\partial u}{\partial x_k} dxdt \\
&\quad + \frac{1}{2}\int_{\Sigma_1} \rho_k\nu_k (a_1 \mid \Delta u_1 \mid^2 - a_2 \mid \Delta u_2 \mid^2) d\Sigma \\
&\quad + \int_{\Sigma_1} a_1 \Delta u_1 \rho_k \frac{\partial^2}{\partial \nu \partial x_k}(u_2 - u_1) d\Sigma.
\end{aligned}
\tag{2.1}
$$

Proof. Multiplying (1.1) by $\rho_k \dfrac{\partial u}{\partial x_k}$ and integrating on Q, we have

$$
\begin{aligned}
&\int_Q \rho_k \frac{\partial u}{\partial x_k} u'' dxdt + \int_Q \rho_k \frac{\partial u}{\partial x_k} a(x)\Delta^2 u dxdt + \int_Q q\rho_k u \frac{\partial u}{\partial x_k} dxdt \\
&= \int_Q \rho_k \frac{\partial u}{\partial x_k} f dxdt.
\end{aligned}
\tag{2.2}
$$

Integrating by parts and noting that $a_1 \dfrac{\partial \Delta u_1}{\partial \nu} = a_2 \dfrac{\partial \Delta u_2}{\partial \nu}$, $\dfrac{\partial u_1}{\partial x_k} = \dfrac{\partial u_2}{\partial x_k}$, $a_1 \Delta u_1 = a_2 \Delta u_2$ on Σ_1, and $\dfrac{\partial u_2}{\partial x_k} = 0$ on Σ, we obtain

$$
\begin{aligned}
\int_Q \rho_k \frac{\partial u}{\partial x_k} u'' dxdt &= \Big(u'(t), \rho_k \frac{\partial u(t)}{\partial x_k}\Big)\Big|_0^T - \frac{1}{2}\int_{\Sigma_1} \rho_k \nu_k \mid u_1' \mid^2 \, d\Sigma \\
&\quad + \frac{1}{2}\int_{Q_1} \frac{\partial \rho_k}{\partial x_k} \mid u_1' \mid^2 \, dxdt + \frac{1}{2}\int_{\Sigma_1} \rho_k \nu_k \mid u_2' \mid^2 \, d\Sigma \\
&\quad + \frac{1}{2}\int_{Q_2} \frac{\partial \rho_k}{\partial x_k} \mid u_2' \mid^2 \, dxdt \\
&= \Big(u'(t), \rho_k \frac{\partial u(t)}{\partial x_k}\Big)\Big|_0^T + \frac{1}{2}\int_Q \frac{\partial \rho_k}{\partial x_k} \mid u' \mid^2 \, dxdt,
\end{aligned}
\tag{2.3}
$$

and

$$\begin{aligned}
&\int_Q a(x)\rho_k \frac{\partial u}{\partial x_k}\Delta^2 u dxdt \\
&= \int_{\Sigma_1} a_1 \frac{\partial \Delta u_1}{\partial \nu}\rho_k \frac{\partial u_1}{\partial x_k} d\Sigma - \int_{\Sigma_1} a_1 \Delta u_1 \frac{\partial}{\partial \nu}\Big(\rho_k \frac{\partial u_1}{\partial x_k}\Big) d\Sigma \\
&\quad + \int_{Q_1} a_1 \Delta u_1 \Delta\Big(\rho_k \frac{\partial u_1}{\partial x_k}\Big) dxdt - \int_{\Sigma_1} a_2 \frac{\partial \Delta u_2}{\partial \nu}\rho_k \frac{\partial u_2}{\partial x_k} d\Sigma \\
&\quad + \int_{\Sigma_1} a_2 \Delta u_2 \frac{\partial}{\partial \nu}\Big(\rho_k \frac{\partial u_2}{\partial x_k}\Big) d\Sigma + \int_{Q_2} a_2 \Delta u_2 \Delta\Big(\rho_k \frac{\partial u_2}{\partial x_k}\Big) dxdt \qquad (2.4) \\
&\quad + \int_{\Sigma} a_2 \frac{\partial \Delta u_2}{\partial \nu}\rho_k \frac{\partial u_2}{\partial x_k} d\Sigma - \int_{\Sigma} a_2 \Delta u_2 \frac{\partial}{\partial \nu}\Big(\rho_k \frac{\partial u_2}{\partial x_k}\Big) d\Sigma \\
&= \int_{\Sigma_1} a_1 \Delta u_1 \rho_k \frac{\partial^2}{\partial \nu \partial x_k}(u_2 - u_1) d\Sigma - \int_{\Sigma} a_2 \Delta u_2 \rho_k \frac{\partial^2 u_2}{\partial \nu \partial x_k} d\Sigma \\
&\quad + \int_{Q_1} a_1 \Delta u_1 \Delta\Big(\rho_k \frac{\partial u_1}{\partial x_k}\Big) dxdt + \int_{Q_2} a_2 \Delta u_2 \Delta\Big(\rho_k \frac{\partial u_2}{\partial x_k}\Big) dxdt.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int_{Q_1} a_1 \Delta u_1 \Delta\Big(\rho_k \frac{\partial u_1}{\partial x_k}\Big) dxdt + \int_{Q_2} a_2 \Delta u_2 \Delta\Big(\rho_k \frac{\partial u_2}{\partial x_k}\Big) dxdt \\
&= \int_Q a(x)[\Delta \rho_k \Delta u \frac{\partial u}{\partial x_k} + 2\frac{\partial \rho_k}{\partial x_j}\Delta u \frac{\partial^2 u}{\partial x_k \partial x_j} - \frac{1}{2}\frac{\partial \rho_k}{\partial x_k} \mid \Delta u \mid^2] dxdt \qquad (2.5) \\
&\quad + \frac{1}{2}\int_{\Sigma} a_2 \rho_k \nu_k \mid \Delta u_2 \mid^2 d\Sigma + \frac{1}{2}\int_{\Sigma_1} \rho_k \nu_k (a_1 \mid \Delta u_1 \mid^2 - a_2 \mid \Delta u_2 \mid^2) d\Sigma.
\end{aligned}$$

Noting that $\dfrac{\partial^2 u_2}{\partial \nu \partial x_k} = \dfrac{\partial^2 u_2}{\partial \nu^2}\nu_k$ and $\dfrac{\partial^2 u_2}{\partial x_k^2} = \dfrac{\partial^2 u_2}{\partial \nu^2}\nu_k^2$ on Σ, it therefore follows from (2.4) that

$$\begin{aligned}
&\int_Q a(x)\rho_k \frac{\partial u}{\partial x_k}\Delta^2 u dxdt \\
&= \int_Q a(x)[\Delta \rho_k \Delta u \frac{\partial u}{\partial x_k} + 2\frac{\partial \rho_k}{\partial x_j}\Delta u \frac{\partial^2 u}{\partial x_k \partial x_j} - \frac{1}{2}\frac{\partial \rho_k}{\partial x_k} \mid \Delta u \mid^2] dxdt \\
&\quad - \frac{1}{2}\int_{\Sigma} a_2 \rho_k \nu_k \mid \Delta u_2 \mid^2 d\Sigma + \int_{\Sigma_1} a_1 \Delta u_1 \rho_k \frac{\partial^2}{\partial \nu \partial x_k}(u_2 - u_1) d\Sigma \qquad (2.6) \\
&\quad + \frac{1}{2}\int_{\Sigma_1} \rho_k \nu_k (a_1 \mid \Delta u_1 \mid^2 - a_2 \mid \Delta u_2 \mid^2) d\Sigma.
\end{aligned}$$

From (2.2), (2.3), and (2.6) we deduce

$$\begin{aligned}
&\int_Q f \rho_k \frac{\partial u}{\partial x_k} dxdt \\
&= \Big(u'(t), \rho_k \frac{\partial u(t)}{\partial x_k}\Big)\Big|_0^T + \frac{1}{2}\int_Q \frac{\partial \rho_k}{\partial x_k} \mid u' \mid^2 dxdt \\
&\quad + \int_Q a(x)[\Delta \rho_k \Delta u \frac{\partial u}{\partial x_k} + 2\frac{\partial \rho_k}{\partial x_j}\Delta u \frac{\partial^2 u}{\partial x_k \partial x_j} - \frac{1}{2}\frac{\partial \rho_k}{\partial x_k} \mid \Delta u \mid^2] dxdt \\
&\quad - \frac{1}{2}\int_{\Sigma} a_2 \rho_k \nu_k \mid \Delta u_2 \mid^2 d\Sigma + \int_{\Sigma_1} a_1 \Delta u_1 \rho_k \frac{\partial^2}{\partial \nu \partial x_k}(u_2 - u_1) d\Sigma \\
&\quad + \frac{1}{2}\int_{\Sigma_1} \rho_k \nu_k (a_1 \mid \Delta u_1 \mid^2 - a_2 \mid \Delta u_2 \mid^2) d\Sigma + \int_Q q \rho_k u \frac{\partial u}{\partial x_k} dxdt.
\end{aligned}$$

This implies (2.1). $\square$

After establishing the identity (2.1), we can obtain the boundary regularity and boundary estimations for the solution of (1.1).

Lemma 2.2. *Suppose the boundary Γ of Ω is of class C^3. Then there exist a constant $c > 0$ such that for all weak solutions of (1.1)*

$$\int_{\Sigma} \mid \Delta u_2 \mid^2 d\Sigma \leq c\Big[\| u^0 \|^2_{2,\Omega} + \| u^1 \|^2_{0,\Omega} + \| f \|^2_{L^1(0,T;L^2(\Omega))}\Big]. \tag{2.7}$$

Proof. We choose $\rho \in [C^2(\overline{\Omega})]^n$ such that $\rho = \nu$ on Γ and $\rho = 0$ in Ω_0, where the open set Ω_0 satisfies $\overline{\Omega}_1 \subset \Omega_0 \subset \overline{\Omega}_0 \subset \Omega$. Then the inequality (2.7) follows from Theorem 1.4 and the identity (2.1). $\square$

Lemma 2.3. *Suppose the boundaries Γ and Γ_1 of Ω and Ω_1 of are class C^4. Assume $q \in W^{1,\infty}(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;W^{2,\infty}(\Omega))$. Then there exist a constant $c > 0$ such that for all strong solutions of (1.1)*

$$\int_{\Sigma} \mid \Delta u_2' \mid^2 d\Sigma \leq c\Big[\|u^0\|^2_{H^4(\Omega,\Gamma_1)} + \| u^1 \|^2_{2,\Omega} + \| f \|^2_{L^1(0,T;H_0^2(\Omega))}\Big]. \tag{2.8}$$

Proof. By density and passage to the limit, it suffices to prove (2.8) for $f \in \mathcal{D}((0,T);\ H_0^2(\Omega))$, where $\mathcal{D}((0,T);H_0^2(\Omega))$ denotes the space of all infinitely differentiable functions in t with compact supports in $(0,T)$ and values in $H_0^2(\Omega)$.

Let u be the solution of (1.1) and set $v = u'$, then v satisfies

$$\begin{cases} v'' + a(x)\Delta^2 v + qv + q'[u^0 + \int_0^t v(s)ds] = f' & \text{in } Q, \\ v(0) = u^1, \quad v'(0) = -a(x)\Delta^2 u^0 - qu^0 & \text{in } \Omega, \\ v_2 = \dfrac{\partial v_2}{\partial \nu} = 0, & \text{on } \Sigma, \\ v_1 = v_2, \quad \dfrac{\partial v_1}{\partial \nu} = \dfrac{\partial v_2}{\partial \nu} & \text{on } \Sigma_1, \\ a_1 \Delta v_1 = a_2 \Delta v_2, \quad a_1 \dfrac{\partial \Delta v_1}{\partial \nu} = a_2 \dfrac{\partial \Delta v_2}{\partial \nu} & \text{on } \Sigma_1. \end{cases} \tag{2.9}$$

Choosing a vector field ρ as in the proof of Lemma 2.2 and using the identity (2.1) (at this time, we have a more term $\displaystyle\int_Q q'[u^0 + \int_0^t v(s)ds]\rho_k \frac{\partial v}{\partial x_k} dxdt$), we obtain

$$\begin{aligned} &\frac{1}{2}\int_{\Sigma} a_2 \mid \Delta u_2' \mid^2 \ d\Sigma \\ &= \frac{1}{2}\int_{\Sigma} a_2 \mid \Delta v_2 \mid^2 \ d\Sigma \\ &= \Big(v'(t), \rho_k \frac{\partial v(t)}{\partial x_k}\Big)\Big|_0^T + \frac{1}{2}\int_Q \frac{\partial \rho_k}{\partial x_k}[\mid v' \mid^2 - a(x) \mid \Delta v \mid^2] dxdt \\ &\quad + \int_Q a(x)\Delta\rho_k \Delta v \frac{\partial v}{\partial x_k} \ dxdt + 2\int_Q a(x)\frac{\partial \rho_k}{\partial x_j}\Delta v \frac{\partial^2 v}{\partial x_k \partial x_j} \ dxdt \\ &\quad - \int_Q f' \rho_k \frac{\partial v}{\partial x_k} \ dxdt + \int_Q q\rho_k v \frac{\partial v}{\partial x_k} \ dxdt \\ &\quad + \int_Q q'[u^0 + \int_0^t v(s)ds]\rho_k \frac{\partial v}{\partial x_k} dxdt. \end{aligned} \tag{2.10}$$

Since

$$v' = u'' = f - a(x)\Delta^2 u - qu, \tag{2.11}$$

we deduce

$$\begin{aligned}
&\int_Q \frac{df}{dt}\rho_k \frac{\partial v}{\partial x_k}\ dxdt \\
&= -\int_Q f\rho_k \frac{\partial v'}{\partial x_k}\ dxdt \\
&= \int_Q v' \frac{\partial}{\partial x_k}(f\rho_k)\ dxdt \\
&= \int_Q f \frac{\partial}{\partial x_k}(f\rho_k)\ dxdt - \int_Q (a(x)\Delta^2 u + qu)\frac{\partial}{\partial x_k}(f\rho_k)\ dxdt \\
&= \int_Q f^2 \frac{\partial \rho_k}{\partial x_k}\ dxdt + \frac{1}{2}\int_Q \rho_k \frac{\partial}{\partial x_k}(f^2)\ dxdt \\
&\quad - \int_Q (a(x)\Delta^2 u + qu)\frac{\partial}{\partial x_k}(f\rho_k)\ dxdt \\
&= \frac{1}{2}\int_Q f^2 \frac{\partial \rho_k}{\partial x_k}\ dxdt - \int_Q (a(x)\Delta^2 u + qu)\frac{\partial}{\partial x_k}(f\rho_k)\ dxdt.
\end{aligned} \tag{2.12}$$

Substituting (2.11) and (2.12) into (2.10) yields

$$\begin{aligned}
&\frac{1}{2}\int_\Sigma a_2 \mid \Delta u_2' \mid^2\ d\Sigma \\
&= \Big(v'(t), \rho_k \frac{\partial v(t)}{\partial x_k}\Big)\Big|_0^T + \frac{1}{2}\int_Q \frac{\partial \rho_k}{\partial x_k}[\mid f - a(x)\Delta^2 u - qu \mid^2 - a(x) \mid \Delta v \mid^2] dxdt \\
&\quad + \int_Q a(x)\Delta\rho_k \Delta v \frac{\partial v}{\partial x_k}\ dxdt + 2\int_Q a(x)\frac{\partial \rho_k}{\partial x_j}\Delta v \frac{\partial^2 v}{\partial x_k \partial x_j}\ dxdt \\
&\quad - \frac{1}{2}\int_Q f^2 \frac{\partial \rho_k}{\partial x_k}\ dxdt + \int_Q (a(x)\Delta^2 u + qu)\frac{\partial}{\partial x_k}(f\rho_k)\ dxdt \\
&\quad + \int_Q q\rho_k v \frac{\partial v}{\partial x_k}\ dxdt + \int_Q q'[u^0 + \int_0^t v(s)ds]\rho_k \frac{\partial v}{\partial x_k} dxdt \\
&= \Big(v'(t), \rho_k \frac{\partial v(t)}{\partial x_k}\Big)\Big|_0^T + \int_Q a(x)\Delta\rho_k \Delta v \frac{\partial v}{\partial x_k}\ dxdt \\
&\quad + 2\int_Q a(x)\frac{\partial \rho_k}{\partial x_j}\Delta v \frac{\partial^2 v}{\partial x_k \partial x_j}\ dxdt \\
&\quad + \frac{1}{2}\int_Q \frac{\partial \rho_k}{\partial x_k}[a^2(x) \mid \Delta^2 u \mid^2 + 2aqu\Delta^2 u + q^2 u^2 - a(x) \mid \Delta v \mid^2]\ dxdt \\
&\quad + \int_Q (a(x)\Delta^2 u + qu)\rho_k \frac{\partial f}{\partial x_k}\ dxdt + \int_Q q\rho_k v \frac{\partial v}{\partial x_k}\ dxdt \\
&\quad + \int_Q q'[u^0 + \int_0^t v(s)ds]\rho_k \frac{\partial v}{\partial x_k} dxdt.
\end{aligned} \tag{2.13}$$

Since $f(0) = f(T) = 0$, it follows from (2.13) and Theorem 1.4 that

$$\int_\Sigma \mid \Delta u_2' \mid^2 d\Sigma \le c\Big[\|u^0\|^2_{H^4(\Omega,S)} + \| u^1 \|^2_{2,\Omega} + \| f \|^2_{L^1(0,T;H_0^2(\Omega))}\Big]. \qquad \square$$

§3. Nonhomogeneous Boundary Value Problems

We now are in a position to consider the nonhomogeneous boundary value problem which introduces a control function $\phi \in L^2(\Sigma)$:

$$\begin{cases} u'' + a(x)\Delta^2 u + qu = 0 & \text{in } Q, \\ u(0) = u^0, \quad u'(0) = u^1 & \text{in } \Omega, \\ u_2 = 0, \quad \dfrac{\partial u_2}{\partial \nu} = \phi & \text{on } \Sigma, \\ u_1 = u_2, \quad \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} & \text{on } \Sigma_1, \\ a_1 \Delta u_1 = a_2 \Delta u_2, \quad a_1 \dfrac{\partial \Delta u_1}{\partial \nu} = a_2 \dfrac{\partial \Delta u_2}{\partial \nu} & \text{on } \Sigma_1. \end{cases} \tag{3.1}$$

The solution of (3.1) can be defined by the transposition method (see [12]) as follows.

Definition 3.1. *u is said to be a ultraweak solution of (3.1) for $(u^0, u^1) \in L^2(\Omega) \times H^{-2}(\Omega)$ if we have*

$$\int_Q f u dx dt = \langle \theta(0), u^1 \rangle - \int_\Omega \theta'(0) u^0 \ dx - \int_\Sigma a_2 \phi \Delta \theta_2 d\Sigma, \qquad \forall f \in \mathcal{D}(Q), \tag{3.2}$$

where $\theta = \theta(x,t)$ is the solution of

$$\begin{cases} \theta'' + a(x)\Delta^2 \theta + q\theta = f & \text{in } Q, \\ \theta(T) = \theta'(T) = 0 & \text{in } \Omega, \\ \theta_2 = \dfrac{\partial \theta_2}{\partial \nu} = 0 & \text{on } \Sigma, \\ \theta_1 = \theta_2, \quad \dfrac{\partial \theta_1}{\partial \nu} = \dfrac{\partial \theta_2}{\partial \nu} & \text{on } \Sigma_1, \\ a_1 \Delta \theta_1 = a_2 \Delta \theta_2, \quad a_1 \dfrac{\partial \Delta \theta_1}{\partial \nu} = a_2 \dfrac{\partial \Delta \theta_2}{\partial \nu} & \text{on } \Sigma_1. \end{cases} \tag{3.3}$$

and $\langle \cdot, \cdot \rangle$ denotes the dual product between the spaces $H_0^2(\Omega)$ and $H^{-2}(\Omega)$, and $\mathcal{D}(Q)$ denotes the space of all infinitely differentiable functions defined on Q with compact supports in Q.

Theorem 3.2. *Suppose the boundaries Γ and Γ_1 are of class C^4. Assume $q \in W^{1,\infty}(0,T; W^{2,\infty}(\Omega))$. Then for all $(u^0, u^1, \phi) \in L^2(\Omega) \times H^{-2}(\Omega) \times L^2(\Sigma)$, there exists a unique ultraweak solution of (3.1) with*

$$u \in C([0,T]; L^2(\Omega)) \times C^1([0,T]; H^{-2}(\Omega)). \tag{3.4}$$

Moreover, there exists $c > 0$ such that for all $t \in [0,T]$

$$\| u(t) \|_{0,\Omega} + \| u'(t) \|_{-2,\Omega} \leq c[\| u^0 \|_{0,\Omega} + \| u^1 \|_{-2,\Omega} + \| \phi \|_{L^2(\Sigma)}]. \tag{3.5}$$

Corollary 3.3. *Let Γ and Γ_1 and q be as in Theorem 3.2. Then for all $(u^0, u^1, \phi, f) \in L^2(\Omega) \times H^{-2}(\Omega) \times L^2(\Sigma) \times L^1(0,T; L^2(\Omega))$, there exists a unique ultraweak solution of*

$$\begin{cases} u'' + a(x)\Delta^2 u + qu = f & \text{in } Q, \\ u(0) = u^0, \quad u'(0) = u^1 & \text{in } \Omega, \\ u_2 = 0, \quad \dfrac{\partial u_2}{\partial \nu} = \phi & \text{on } \Sigma, \\ u_1 = u_2, \quad \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} & \text{on } \Sigma_1, \\ a_1 \Delta u_1 = a_2 \Delta u_2, \quad a_1 \dfrac{\partial \Delta u_1}{\partial \nu} = a_2 \dfrac{\partial \Delta u_2}{\partial \nu} & \text{on } \Sigma_1, \end{cases} \tag{3.6}$$

with

$$u \in C([0,T]; L^2(\Omega)) \times C^1([0,T]; H^{-2}(\Omega)). \tag{3.7}$$

Moreover, there exists $c > 0$ *such thatfor all* $t \in [0,T]$

$$\| u(t) \|_{0,\Omega} + \| u'(t) \|_{-2,\Omega} \leq \tag{3.8}$$

$$c[\| u^0 \|_{0,\Omega} + \| u^1 \|_{-2,\Omega} + \| \phi \|_{L^2(\Sigma)} + \| f \|_{L^1(0,T;L^2(\Omega))}].$$

To prove theorem 3.2, the following lemmas are necessary.

Lemma 3.4. *Suppose the boundaries* Γ *and* Γ_1 *are of class* C^4. *Assume* $q \in W^{1,\infty}(0,T; W^{2,\infty}(\Omega))$. *If* $f = \dfrac{dg}{dt}$ *with* $g \in L^1(0,T; H_0^2(\Omega))$, *then there exists a constant* $c > 0$ *such that for all solutions of (1.1) with* $u^0 = u^1 = 0$

$$\| u(T) \|_{2,\Omega} + \| u'(T) \|_{0,\Omega} + \| \Delta u_2 \|_{L^2(\Sigma)} \leq c \| g \|_{L^1(0,T;H_0^2(\Omega))} . \tag{3.9}$$

Proof. By density and passage to the limit, it suffices to prove (3.9) for $g \in \mathcal{D}((0,T);\ H_0^2(\Omega))$.

Let w be the solution of

$$\begin{cases} w'' + a(x)\Delta^2 w + qw - \int_0^t q'(x,s)w(x,s)ds = g & \text{in } Q, \\ w(0) = w'(0) = 0 & \text{in } \Omega, \\ w_2 = \dfrac{\partial w_2}{\partial \nu} = 0 & \text{on } \Sigma, \\ w_1 = w_2, \quad \dfrac{\partial w_1}{\partial \nu} = \dfrac{\partial w_2}{\partial \nu} & \text{on } \Sigma_1, \\ a_1 \Delta w_1 = a_2 \Delta w_2, \quad a_1 \dfrac{\partial \Delta w_1}{\partial \nu} = a_2 \dfrac{\partial \Delta w_2}{\partial \nu} & \text{on } \Sigma_1. \end{cases} \tag{3.10}$$

Then $u = w'$ is the solution of (1.1) with $u^0 = u^1 = 0$ since $u'(0) = w''(0) = g(0) - a(x)\Delta^2 w(0) - qw(0) = 0$(note that $g(0) = 0$). (3.10) is solvable because $w = \int_0^t u(s)ds$. Similar to (1.12) and (1.14), we deduce that there is a constant $c > 0$ such that

$$\| w(t) \|_{2,\Omega} + \| w'(t) \|_{0,\Omega} \leq c \| g \|_{L^1(0,T;L^2(\Omega))}, \qquad \forall t \in [0,T], \tag{3.11}$$

$$\| \Delta w'(t) \|_{0,\Omega} + \| \Delta^2 w(t) \|_{0,\Omega} \leq c \| g \|_{L^1(0,T;H_0^2(\Omega))}, \qquad \forall t \in [0,T]. \tag{3.12}$$

Since

$$u = w' \text{ and } u'(T) = w''(T) = -a(x)\Delta^2 w(T) - qw(T) + \int_0^T q'(x,s)w(x,s)ds,$$

(because $g(T) = 0$), it follows from (3.11) and (3.12) that

$$\| u(t) \|_{2,\Omega} + \| \Delta^2 w(t) \|_{0,\Omega} \leq c \| g \|_{L^1(0,T;H_0^2(\Omega))}, \qquad \forall t \in [0,T], \tag{3.13}$$

and

$$\begin{aligned} &\| u(T) \|_{2,\Omega} + \| u'(T) \|_{0,\Omega} \\ &\leq c(\| g \|_{L^1(0,T;H_0^2(\Omega))} + \|w(T) + \int_0^T q'(x,s)w(x,s)ds\|_{0,\Omega}) \\ &\leq c \| g \|_{L^1(0,T;H_0^2(\Omega))} . \end{aligned} \tag{3.14}$$

On the other hand, choosing $\rho \in [C^2(\overline{\Omega})]^n$ such that $\rho = \nu$ on Γ and $\rho = 0$ in Ω_0, where the open set Ω_0 satisfies $\overline{\Omega}_1 \subset \Omega_0 \subset \overline{\Omega}_0 \subset \Omega$, and applying (2.1), we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_\Sigma a_2 \mid \Delta u_2 \mid^2 \ d\Sigma \\
=&\Big(u'(T), \rho_k \frac{\partial u(T)}{\partial x_k}\Big) + \frac{1}{2}\int_Q \frac{\partial \rho_k}{\partial x_k}[\mid u' \mid^2 - a(x) \mid \Delta u \mid^2] dxdt \\
&+ 2\int_Q a(x)\frac{\partial \rho_k}{\partial x_j}\Delta u \frac{\partial^2 u}{\partial x_k \partial x_j} \ dxdt + \int_Q a(x)\Delta \rho_k \Delta u \frac{\partial u}{\partial x_k} \ dxdt \\
&- \int_Q \frac{dg}{dt}\rho_k \frac{\partial u}{\partial x_k} \ dxdt + \int_Q q\rho_k u \frac{\partial u}{\partial x_k} \ dxdt.
\end{aligned}
\tag{3.15}
$$

Since

$$
u' = w'' = g - a(x)\Delta^2 w - qw + \int_0^t q'(x,s)w(x,s)ds, \tag{3.16}
$$

we deduce in the same manner as in (2.12)

$$
\begin{aligned}
&\int_Q \frac{dg}{dt}\rho_k \frac{\partial u}{\partial x_k} \ dxdt \\
=& \frac{1}{2}\int_Q g^2 \frac{\partial \rho_k}{\partial x_k} \ dxdt - \int_Q (a(x)\Delta^2 w + qw)\frac{\partial}{\partial x_k}(g\rho_k) \ dxdt \\
&+ \int_Q \frac{\partial}{\partial x_k}(g\rho_k)\int_0^t q'(x,s)w(x,s)dsdxdt.
\end{aligned}
\tag{3.17}
$$

Substituting (3.16) and (3.17) into (3.15) yields

$$
\begin{aligned}
&\frac{1}{2}\int_\Sigma a_2 \mid \Delta u_2 \mid^2 \ d\Sigma \\
=&\Big(u'(T), \rho_k \frac{\partial u(T)}{\partial x_k}\Big) + \int_Q a(x)\Delta \rho_k \Delta u \frac{\partial u}{\partial x_k} \ dxdt + \int_Q q\rho_k u \frac{\partial u}{\partial x_k} \ dxdt \\
&+ \frac{1}{2}\int_Q \frac{\partial \rho_k}{\partial x_k}[a^2(x) \mid \Delta^2 w \mid^2 + 2qaw\Delta^2 w + q^2 w^2 - a(x) \mid \Delta u \mid^2] \ dxdt \\
&+ \int_Q \frac{\partial \rho_k}{\partial x_k}(g - a(x)\Delta^2 w - qw)\int_0^t q'(x,s)w(x,s)dsdxdt \\
&+ \frac{1}{2}\int_Q \frac{\partial \rho_k}{\partial x_k}(\int_0^t q'(x,s)w(x,s)ds)^2 dxdt \\
&+ 2\int_Q a(x)\frac{\partial \rho_k}{\partial x_j}\Delta u \frac{\partial^2 u}{\partial x_k \partial x_j} \ dxdt + \int_Q (a(x)\Delta^2 w + qw)\rho_k \frac{\partial g}{\partial x_k} \ dxdt \\
&- \int_Q \frac{\partial}{\partial x_k}(g\rho_k)\int_0^t q'(x,s)w(x,s)dsdxdt.
\end{aligned}
\tag{3.18}
$$

It therefore follows from (3.13), (3.14), and (3.18) that

$$
\parallel \Delta u_2 \parallel_{L^2(\Sigma)} \leq c \parallel g \parallel_{L^1(0,T;H_0^2(\Omega))} . \tag{3.19}
$$

Finally, (3.9) follows from (3.14) and (3.19). $\square$

Lemma 3.5. *Let Γ and Γ_1 be of class C^4. Then for*

$$\phi_j \in H^{4-j-\frac{1}{2}}(\Gamma) \qquad j = 0, 1, 2, 3 \tag{3.20}$$

there exists $u \in H^4(\Omega_1, \Omega_2)$ such that

$$\frac{\partial^j u}{\partial \nu^j} = \phi_j, \qquad \text{on } \Gamma, \qquad j = 0, 1, 2, 3, \tag{3.21}$$

where

$$H^4(\Omega_1, \Omega_2) = \Big\{ u \ : \ u \in H^2(\Omega); \ u_i \in H^4(\Omega_i), \ i = 1, 2; \tag{3.22}$$

$$a_1 \Delta u_1 = a_2 \Delta u_2 \ \text{and} \ a_1 \frac{\partial \Delta u_1}{\partial \nu} = a_2 \frac{\partial \Delta u_2}{\partial \nu} \ \text{on } \Gamma_1 \Big\}.$$

Proof. By the trace theorem (see [12], Chapter 1), it follows that there exists $w \in H^4(\Omega)$ such that

$$\frac{\partial^j w}{\partial \nu^j} = \phi_j, \qquad \text{on } \Gamma, \qquad j = 0, 1, 2, 3. \tag{3.23}$$

Likewise, there exists $v \in H^4(\Omega_1)$ such that

$$v = w \qquad \text{on } \Gamma_1, \tag{3.24}$$

$$\frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} \qquad \text{on } \Gamma_1, \tag{3.25}$$

$$\Delta v = \frac{a_2}{a_1} \Delta w \qquad \text{on } \Gamma_1, \tag{3.26}$$

$$\frac{\partial \Delta v}{\partial \nu} = \frac{a_2}{a_1} \frac{\partial \Delta w}{\partial \nu} \qquad \text{on } \Gamma_1. \tag{3.27}$$

Then, u defined by

$$u = \begin{cases} v, & x \in \Omega_1, \\ w, & x \in \Omega_2, \end{cases} \tag{3.28}$$

belongs to $H^4(\Omega_1, \Omega_2)$ and satisfies

$$\frac{\partial^j u}{\partial \nu^j} = \phi_j, \qquad \text{on } \Gamma, \qquad j = 0, 1, 2, 3. \tag{3.29}$$

□

We now are ready to prove Theorem 3.2.

Proof of Theorem 3.2: It follows from Definition 3.1, Theorem 1.4, and Lemma 2.2 that

$$\begin{aligned} | \int_Q uf \ dxdt \ | \leq c[\| \ u^0 \ \|_{0,\Omega} + \| \ u^1 \ \|_{-2,\Omega} + \| \ \phi \ \|_{L^2(\Sigma)}] \times \\ \times [\| \ \theta(0) \ \|_{2,\Omega} + \| \ \theta'(0) \ \|_{0,\Omega} + \| \ \Delta\theta_2 \ \|_{L^2(\Sigma)}] \\ \leq c[\| \ u^0 \ \|_{0,\Omega} + \| \ u^1 \ \|_{-2,\Omega} + \| \ \phi \ \|_{L^2(\Sigma)}] \ \| \ f \ \|_{L^1(0,T;L^2(\Omega))} \ . \end{aligned} \tag{3.30}$$

Therefore, there exists a $u \in L^\infty(0, T; L^2(\Omega))$ such that (3.2) holds. Moreover,

$$\| \ u \ \|_{L^\infty(0,T;L^2(\Omega))} \leq c[\| \ u^0 \ \|_{0,\Omega} + \| \ u^1 \ \|_{-2,\Omega} + \| \ \phi \ \|_{L^2(\Sigma)}]. \tag{3.31}$$

On the other hand, if $f = \dfrac{dg}{dt}$ with $g \in L^1(0,T;H_0^2(\Omega))$, then Lemma 3.4 gives

$$| \int_Q u\frac{dg}{dt}\, dxdt \,| \leq c[\| u^0 \|_{0,\Omega} + \| u^1 \|_{-2,\Omega} + \| \phi \|_{L^2(\Sigma)}] \| g \|_{L^1(0,T;H_0^2(\Omega))}, \tag{3.32}$$

which implies $u' \in L^\infty(0,T;H^{-2}(\Omega))$, and

$$\| u' \|_{L^\infty(0,T;H^{-2}(\Omega))} \leq c[\| u^0 \|_{0,\Omega} + \| u^1 \|_{-2,\Omega} + \| \phi \|_{L^2(\Sigma)}]. \tag{3.33}$$

Hence, if we can prove

$$u \in C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-2}(\Omega)) \tag{3.34}$$

for sufficiently regular u^0, u^1, and ϕ, then by the density, the theorem follows from (3.31) and (3.33). In doing so, we may as well assume that $(u^0,u^1) \in H_0^2(\Omega) \times L^2(\Omega)$ and $\phi \in \mathcal{D}((0,T);H^{\frac{5}{2}}(\Gamma))$. By Lemma 3.5, there is $\psi \in \mathcal{D}((0,T);H^4(\Omega_1,\Omega_2) \cap H_0^1(\Omega))$ such that

$$\frac{\partial \psi}{\partial \nu} = \phi, \qquad \text{on } \Sigma. \tag{3.35}$$

Set

$$v = u - \psi, \tag{3.36}$$

then

$$\begin{cases} v'' + a(x)\Delta^2 v + qv = -(\psi'' + a(x)\Delta^2\psi + q\psi) = F & \text{in } Q, \\ v(0) = u^0, \quad v'(0) = u^1 & \text{in } \Omega, \\ v_2 = \dfrac{\partial v_2}{\partial \nu} = 0 & \text{on } \Sigma, \\ v_1 = v_2, \quad \dfrac{\partial v_1}{\partial \nu} = \dfrac{\partial v_2}{\partial \nu} & \text{on } \Sigma_1, \\ a_1\Delta v_1 = a_2\Delta v_2, \quad a_1\dfrac{\partial \Delta v_1}{\partial \nu} = a_2\dfrac{\partial \Delta v_2}{\partial \nu} & \text{on } \Sigma_1. \end{cases} \tag{3.37}$$

It is clear that $F \in L^1(0,T;L^2(\Omega))$. Thus, it follows from Theorem 1.4 that

$$v \in C([0,T];H_0^2(\Omega)) \cap C^1([0,T];L^2(\Omega)). \tag{3.38}$$

Consequently,

$$\begin{aligned} u = v + \psi &\in C([0,T];H^2(\Omega)) \cap C^1([0,T];L^2(\Omega)) \\ &\subset C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-2}(\Omega)). \end{aligned} \tag{3.39}$$

□

We now consider the nonhomogeneous boundary value problem (3.1) in the case where $\phi \in (H^1(0,T;L^2(\Gamma))'$, i.e., $\phi = \dfrac{\partial \xi}{\partial t}$ with $\xi \in L^2(\Sigma)$. Therefore problem (3.1) now becomes

$$\begin{cases} u'' + a(x)\Delta^2 u + qu = 0 & \text{in } Q, \\ u(0) = u^0, \quad u'(0) = u^1 & \text{in } \Omega, \\ u_2 = 0, \quad \dfrac{\partial u_2}{\partial \nu} = \dfrac{\partial \xi}{\partial t} & \text{on } \Sigma, \\ u_1 = u_2, \quad \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} & \text{on } \Sigma_1, \\ a_1\Delta u_1 = a_2\Delta u_2, \quad a_1\dfrac{\partial \Delta u_1}{\partial \nu} = a_2\dfrac{\partial \Delta u_2}{\partial \nu} & \text{on } \Sigma_1. \end{cases} \tag{3.40}$$

This problem can be solved in a manner similar to the above. To this end, let $(\theta^0, \theta^1, f) \in H^4(\Omega, \Gamma_1) \times H_0^2(\Omega) \times L^1(0, T; H_0^2(\Omega))$ and $\theta = \theta(x,t)$ be the solution of

$$\begin{cases} \theta'' + a(x)\Delta^2\theta + q\theta = f & \text{in } Q, \\ \theta(T) = \theta^0, \quad \theta'(T) = \theta^1 & \text{in } \Omega, \\ \theta_2 = \dfrac{\partial \theta_2}{\partial \nu} = 0 & \text{on } \Sigma, \\ \theta_1 = \theta_2, \quad \dfrac{\partial \theta_1}{\partial \nu} = \dfrac{\partial \theta_2}{\partial \nu} & \text{on } \Sigma_1, \\ a_1\Delta\theta_1 = a_2\Delta\theta_2, \quad a_1\dfrac{\partial \Delta\theta_1}{\partial \nu} = a_2\dfrac{\partial \Delta\theta_2}{\partial \nu} & \text{on } \Sigma_1. \end{cases} \tag{3.41}$$

Multiplying the first equation of (3.41) by u and formally integrating by parts over Q, we obtain

$$\begin{aligned} &\int_Q fudxdt + \langle \theta^0, u'(T)\rangle - \langle \theta^1, u(T)\rangle \\ &= \langle \theta(0), u^1\rangle - \langle \theta'(0), u^0\rangle - \int_\Sigma a_2\frac{\partial \xi}{\partial t}\Delta\theta_2 d\Sigma. \end{aligned} \tag{3.42}$$

Here and in the sequel $\langle \cdot, \cdot \rangle$ denote various dual products between a space and its dual such as $H_0^2(\Omega)$ and $H^{-2}(\Omega)$. On the other hand, in the sense of derivatives in $(H^1(0,T; L^2(\Gamma))'$, we have

$$\int_\Sigma a_2\frac{\partial \xi}{\partial t}\Delta\theta_2 d\Sigma = -\int_\Sigma a_2\xi\Delta\theta_2' d\Sigma. \tag{3.43}$$

It therefore follows that

$$\begin{aligned} &\int_Q fudxdt + \langle \theta^0, u'(T)\rangle - \langle \theta^1, u(T)\rangle \\ &= \langle \theta(0), u^1\rangle - \langle \theta'(0), u^0\rangle + \int_\Sigma a_2\xi\Delta\theta_2' d\Sigma. \end{aligned} \tag{3.44}$$

Set

$$L(\theta^0, \theta^1, f) = \langle \theta(0), u^1\rangle - \langle \theta'(0), u^0\rangle + \int_\Sigma a_2\xi\Delta\theta_2' d\Sigma. \tag{3.45}$$

It follows from Theorem 1.4 and Lemma 2.3 that

$$\begin{aligned} |\, L(\theta^0, \theta^1, f) \,| &\le \|\theta(0)\|_{H^4(\Omega,\Gamma_1)}\|u^1\|_{(H^4(\Omega,\Gamma_1))'} + \|\, \theta'(0) \,\|_{2,\Omega}\|\, u^0 \,\|_{-2,\Omega} \\ &\quad + a_2 \|\, \xi \,\|_{L^2(\Sigma)} \times \|\, \Delta\theta_2' \,\|_{L^2(\Sigma)} \\ &\le c[\|\, u^0 \,\|_{-2,\Omega} + \|u^1\|_{(H^4(\Omega,\Gamma_1))'} + \|\, \xi \,\|_{L^2(\Sigma)}] \\ &\quad \times [\|\theta^0\|_{H^4(\Omega,\Gamma_1)} + \|\, \theta^1 \,\|_{2,\Omega} + \|\, f \,\|_{L^1(0,T;H_0^2(\Omega))}]. \end{aligned} \tag{3.46}$$

This shows that L is a linear continuous functional on $H^4(\Omega, \Gamma_1) \times H_0^2(\Omega) \times L^1(0, T; H_0^2(\Omega))$. Hence there exists a $u \in L^\infty(0, T; H^{-2}(\Omega))$ and $(\psi^0, \psi^1) \in H^{-2}(\Omega) \times (H^4(\Omega, \Gamma_1))'$ such that

$$\langle u, f\rangle + \langle \psi^1, \theta^0\rangle + \langle -\psi^0, \theta^1\rangle = L(\theta^0, \theta^1, f). \tag{3.47}$$

for all $(\theta^0, \theta^1, f) \in H^4(\Omega, \Gamma_1) \times H_0^2(\Omega) \times L^1(0, T; H_0^2(\Omega)$. Moreover,

$$\|\, u \,\|_{L^\infty(0,T;H^{-2}(\Omega))} + \|\, \psi^0 \,\|_{-2,\Omega} + \|\psi^1\|_{(H^4(\Omega,\Gamma_1))'} \tag{3.48}$$

$$\le C[\|\, u^0 \,\|_{-2,\Omega} + \|u^1\|_{(H^4(\Omega,\Gamma_1))'} + \|\, \xi \,\|_{L^2(\Sigma)}].$$

The above calculations lead to

Definition 3.6. *The function u that satisfies (3.47) is said to be a ultraweak solution of the problem (3.40), and ψ^0, ψ^1 that satisfy (3.47) are defined to be values of u, u' at T, respectively, i.e.,*

$$u(T) = \psi^0, \qquad u'(T) = \psi^1. \tag{3.49}$$

Of course, we need to legitimatize the definition 3.6 a little bit. We do so in the case that q is independent of t. The following procedure is not applicable for the case that q depends on t. We can arrive at this point by carefully choosing particular "test functions" θ^0, θ^1, and f. In doing so, we introduce the eigenfunction w of $a(x)\Delta^2 + q$:

$$\begin{cases} a(x)\Delta^2 w + qw = \lambda w & \text{in } \Omega, \\ w_2 = \dfrac{\partial w_2}{\partial \nu} = 0 & \text{on } \Gamma, \\ w_1 = w_2, \quad \dfrac{\partial w_1}{\partial \nu} = \dfrac{\partial w_2}{\partial \nu} & \text{on } \Gamma_1, \\ a_1 \Delta w_1 = a_2 \Delta w_2, \quad a_1 \dfrac{\partial \Delta w_1}{\partial \nu} = a_2 \dfrac{\partial \Delta w_2}{\partial \nu} & \text{on } \Gamma_1. \end{cases} \tag{3.50}$$

We take

$$f = h(t)w. \tag{3.51}$$

Then

$$\theta = p(t)w \tag{3.52}$$

with

$$p'' + \lambda p = h. \tag{3.53}$$

Substituting (3.51) and (3.52) into (3.47), we obtain

$$\begin{aligned} &\int_0^T \langle u, w\rangle(p'' + \lambda p)\ dt + \langle \psi^1, w\rangle p(T) - \langle \psi^0, w\rangle p'(T) \\ &= \langle u^1, w\rangle p(0) - \langle u^0, w\rangle p'(0) + \int_\Sigma a_2 \xi p' \Delta w_2 d\Sigma. \end{aligned} \tag{3.54}$$

Taking $p(0) = p'(0) = p(T) = p'(T) = 0$, we get

$$\langle u, w\rangle'' + \lambda \langle u, w\rangle = -\int_\Gamma a_2 \xi' \Delta w_2 d\Gamma, \tag{3.55}$$

where derivatives $\langle u, w\rangle''$ and ξ' are understood in the sense of duals $(H^2\ (0,T))'$ and $(H^1(0,T); L^2(\Gamma))'$, respectively. It therefore follows from (3.54) that

$$\begin{aligned} &\langle u, w\rangle p'|_0^T - \langle u, w\rangle' p|_0^T + \langle \psi^1, w\rangle p(T) - \langle \psi^0, w\rangle p'(T) \\ &= \langle u^1, w\rangle p(0) - \langle u^0, w\rangle p'(0). \end{aligned} \tag{3.56}$$

By taking $p(0) \neq 0$ and $p'(0) = p(T) = p'(T) = 0$, we get

$$\langle u^1, w\rangle = \langle u, w\rangle'(0). \tag{3.57}$$

Likewise, we obtain

$$\langle u^0, w\rangle = \langle u, w\rangle(0), \quad \langle \psi^0, w\rangle = \langle u, w\rangle(T), \quad \langle \psi^1, w\rangle = \langle u, w\rangle'(T). \tag{3.58}$$

Thus

$$u(0) = u^0, \quad u'(0) = u^1, \quad u(T) = \psi^0, \quad u'(T) = \psi^1$$

hold in the sense of (3.57) and (3.58).

In addition, as in the proof of Theorem 3.2, by a density argument, we can show that

$$u \in C([0,T]; H^{-2}(\Omega)). \tag{3.59}$$

So far we have proved the following.

Theorem 3.7. *Suppose the boundaries Γ and Γ_1 are of class C^4. Assume $q \in W^{1,\infty}(0,T; W^{2,\infty}(\Omega))$. Then for all $(u^0, u^1, \xi) \in H^{-2}(\Omega) \times (H^4(\Omega, \Gamma_1))' \times L^2(\Sigma)$, there exists a unique ultraweak solution of (3.40) in the sense of the definition 3.6 with*

$$u \in C([0,T]; H^{-2}(\Omega)), \tag{3.60}$$

$$u(T) \in H^{-2}(\Omega), \qquad u'(T) \in (H^4(\Omega, \Gamma_1))'. \tag{3.61}$$

Moreover, there exists $c > 0$ such that

$$\begin{aligned} &\| u \|_{L^\infty(0,T;H^{-2}(\Omega))} + \| \psi^0 \|_{-2,\Omega} + \|\psi^1\|_{(H^4(\Omega,\Gamma_1))'} \\ &\leq c[\| u^0 \|_{-2,\Omega} + \|u^1\|_{(H^4(\Omega,\Gamma_1))'} + \| \xi \|_{L^2(\Sigma)}]. \end{aligned} \tag{3.62}$$

It is known from Lemmas 2.2 and 2.3 that $\Delta u_2|_{L^2(\Sigma)} \in L^2(\Sigma)$ for the solutions u of (1.1) with $(u^0, u^1, f) \in H_0^2(\Omega) \times L^2(\Omega) \times L^1(0,T; L^2(\Omega))$ and $\Delta u_2'|_{L^2(\Sigma)} \in L^2(\Sigma)$ if $(u^0, u^1, f) \in H^4(\Omega, \Gamma_1) \times H_0^2(\Omega) \times L^1(0,T; H_0^2(\Omega))$. We are now interested in the boundary regularity for the ultraweak solutions of (1.1) with $(u^0, u^1, f) \in L^2(\Omega) \times H^{-2}(\Omega) \times L^2(0,T; L^2(\Omega))$. In this case the ultraweak solutions of (1.1) are guaranteed by Corollary 3.3.

Lemma 3.8. *Suppose the boundaries Γ and Γ_1 are of class C^4. Assume $q \in W^{1,\infty}(0,T; W^{2,\infty}(\Omega))$. Then there exist a constant $c > 0$ such that for all ultraweak solutions of (1.1) with $(u^0, u^1, f) \in L^2(\Omega) \times H^{-2}(\Omega) \times L^2(0,T; L^2(\Omega))$*

$$\| \Delta u_2 \|^2_{H^{-1}(0,T;L^2(\Gamma))} \leq c\Big[\| u^1 \|^2_{-2,\Omega} + \| u^0 \|^2_{0,\Omega} + \| f \|^2_{L^2(0,T;L^2(\Omega))}\Big]. \tag{3.63}$$

Proof. We introduce the solution e of

$$\begin{cases} a(x)\Delta^2 e = -u^1 & \text{in } \Omega, \\ e_2 = \dfrac{\partial e_2}{\partial \nu} = 0 & \text{on } \Gamma, \\ e_1 = e_2, \quad \dfrac{\partial e_1}{\partial \nu} = \dfrac{\partial e_2}{\partial \nu} & \text{on } \Gamma_1, \\ a_1 \Delta e_1 = a_2 \Delta e_2, \quad a_1 \dfrac{\partial \Delta e_1}{\partial \nu} = a_2 \dfrac{\partial \Delta e_2}{\partial \nu} & \text{on } \Gamma_1. \end{cases} \tag{3.64}$$

and set

$$w = \int_0^t u(s)ds + e, \quad F = \int_0^t f(s,x)ds + qe - \int_0^t q(x,s)u(s)ds + q\int_0^t u(s)ds, \tag{3.65}$$

where u is the solution of (1.1). It is easily verified that

$$\begin{cases} w'' + a(x)\Delta^2 w + qw = F & \text{in } Q, \\ w(0) = e, \quad w'(0) = u^0 & \text{in } \Omega, \\ w_2 = \dfrac{\partial w_2}{\partial \nu} = 0 & \text{on } \Sigma, \\ w_1 = w_2, \quad \dfrac{\partial w_1}{\partial \nu} = \dfrac{\partial w_2}{\partial \nu} & \text{on } \Sigma_1, \\ a_1 \Delta w_1 = a_2 \Delta w_2, \quad a_1 \dfrac{\partial \Delta w_1}{\partial \nu} = a_2 \dfrac{\partial \Delta w_2}{\partial \nu} & \text{on } \Sigma_1. \end{cases}$$

Since $(e, u^0, F) \in H_0^2(\Omega) \times L^2(\Omega) \times L^1(0, T; L^2(\Omega))$, it follows from Lemma 2.2 that there exist a constant $c > 0$

$$\begin{aligned} \int_\Sigma \mid \Delta w_2 \mid^2 d\Sigma &\leq c\Big[\| e \|_{2,\Omega}^2 + \| u^0 \|_{0,\Omega}^2 + \| F \|_{L^1(0,T;L^2(\Omega))}^2\Big] \\ &\leq c\Big[\| u^1 \|_{-2,\Omega}^2 + \| u^0 \|_{0,\Omega}^2 + \| f \|_{L^2(0,T;L^2(\Omega))}^2\Big]. \end{aligned}$$

Here we have used the fact that $a(x)\Delta^2$ is an isomorphism from $H_0^2(\Omega)$ onto $H^{-2}(\Omega)$. Because $\dfrac{\partial}{\partial t}$ is an isomorphism from $L^2(\Sigma)$ onto $H^{-1}(0, T; L^2(\Gamma))$, we conclude

$$\Delta u_2 = \frac{\partial}{\partial t}(\Delta w_2) \in H^{-1}(0, T; L^2(\Gamma)),$$

and

$$\| \Delta u_2 \|_{H^{-1}(0,T;L^2(\Gamma))}^2 \leq c\Big[\| u^1 \|_{-2,\Omega}^2 + \| u^0 \|_{0,\Omega}^2 + \| f \|_{L^2(0,T;L^2(\Omega))}^2\Big].$$

§4. Observability Inequality

The objective of this section is to establish a priori estimates (the observability inequalities), which will permit us to obtain a uniqueness theorem and, a fortiori, theorems of exact controllability. It will also give supplementary information on the space of controllable initial states.

We define the energy of the solution u of (1.1) by

$$E(u, t) = \frac{1}{2}\int_\Omega [\mid u'(x, t) \mid^2 + a(x) \mid \Delta u \mid^2] dx. \tag{4.1}$$

Then,

$$E(u, t) = E(u, 0) + \int_0^t \int_\Omega (f - qu) u' dx dt. \tag{4.2}$$

Let λ_1 be the smallest eigenvalues of the operator Δ^2 with Dirichlet homogeneous boundary conditions on $L^2(\Omega)$ (see (1.6)). We introduce two constants

$$\mu_1 = \begin{cases} \max\{\lambda_1^{-1},\ \lambda_1^{-1/2}\}, & \text{n=1}, \\ \lambda_1^{-1/2}, & n \geq 2, \end{cases} \tag{4.3}$$

$$R_*^2 = \begin{cases} \max\{R^2(x^0), \frac{3}{4}\}, & \text{n=1}, \\ R^2(x^0), & n \geq 2. \end{cases} \tag{4.4}$$

Lemma 4.1. (The observability inequality) *Suppose the boundary* Γ *of* Ω *is of class* C^3. *Suppose there exists* $x^0 \in \Omega_1$ *such that* $m(x)\cdot\nu(x) \geq 0$ *on* Γ_1, *where* ν *directs towards the exterior of* Ω_1. *Assume* $a_2 \leq a_1$ *and* $T > T(x_0) = \frac{R_*\mu_1}{\sqrt{a_2}}$ *and*

$$q_0 = \max_{(x,t)\in Q} |q(x,t)| < \frac{2(T - R_*\mu_1 a_2^{-1/2})}{T[R_*\lambda_1^{-1/2}a_2^{-1}(3\mu_1 + \lambda_1^{-1/4}) + |n-2|\lambda_1^{-1}a_2^{-1}]}. \tag{4.5}$$

If u *is a weak solution of (1.1) with* $f = 0$, *then*

$$cE(u,0) \leq \frac{a_2 R(x^0)}{2}\int_{\Sigma(x^0)} |\ \Delta u_2\ |^2\, d\Sigma, \tag{4.6}$$

where

$$c = \frac{T[2 - q_0|n-2|\lambda_1^{-1}a_2^{-1} - q_0 R_*\lambda_1^{-1/2}a_2^{-1}(\mu_1 + \lambda_1^{-1/4})]}{1 + q_0 T\lambda_1^{-1/2}a_2^{-1/2}} - 2R_*\mu_1 a_2^{-1/2}. \tag{4.7}$$

Proof. It suffices to prove (4.6) in the case of strong solutions, that is, we assume initial conditions $(u^0, u^1) \in H^4(\Omega,\Gamma_1)\times H_0^2(\Omega)$ because we can pass to the limit in the case of weak solutions.

Taking $\rho_k = m_k$ in Lemma 2.1, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Sigma a_2 m_k\nu_k\ |\ \Delta u_2\ |^2\, d\Sigma \\
=&\Big(u'(t), m_k\frac{\partial u(t)}{\partial x_k}\Big)\Big|_0^T + 2\int_0^T E(t)dt \\
&+ \frac{n-2}{2}\int_Q\Big(\ |\ u'\ |^2 - a(x)\ |\ \nabla u\ |^2\Big)dxdt + \int_Q q m_k u\frac{\partial u}{\partial x_k}dxdt \\
&+ \int_{\Sigma_1} a_1\Delta u_1 m_k\frac{\partial^2}{\partial\nu\partial x_k}(u_2 - u_1)d\Sigma \\
&+ \frac{1}{2}\int_{\Sigma_1} m_k\nu_k(a_1\ |\ \Delta u_1\ |^2 - a_2\ |\ \Delta u_2\ |^2)d\Sigma.
\end{aligned} \tag{4.8}$$

On the other hand, multiplying the first equation of (1.1) by u and integrating over Q, we obtain

$$\begin{aligned}
&(u', u)\Big|_0^T - \int_Q |\ u'\ |^2 + \int_{\Sigma_1} a_1\frac{\partial\Delta u_1}{\partial\nu}u_1 d\Sigma - \int_{\Sigma_1} a_1\Delta u_1\frac{\partial u_1}{\partial\nu}d\Sigma \\
&\quad + \int_{Q_1} a_1\ |\ \Delta u_1\ |^2\ dxdt - \int_{\Sigma_1} a_2\frac{\partial\Delta u_2}{\partial\nu}u_2 d\Sigma + \int_{\Sigma_1} a_2\Delta u_2\frac{\partial u_2}{\partial\nu}d\Sigma \\
&\quad + \int_{Q_2} a_2\ |\ \Delta u_2\ |^2\ dxdt + \int_Q q\ |\ u\ |^2\ dxdt \\
&= 0.
\end{aligned}$$

The transmission condition gives

$$(u'(t), u(t))\Big|_0^T + \int_Q q\ |\ u\ |^2\ dxdt = \int_Q\Big(|\ u'\ |^2 - a(x)\ |\ \Delta u\ |^2\Big)dxdt. \tag{4.9}$$

Therefore, (4.8) becomes

$$
\begin{aligned}
&\frac{1}{2}\int_{\Sigma} a_2 m_k \nu_k \mid \Delta u_2 \mid^2 d\Sigma \\
=&\Big(u'(t), m_k\frac{\partial u(t)}{\partial x_k}+\frac{n-2}{2}u(t)\Big)\Big|_0^T+2\int_0^T E(t)dt \\
&+\frac{n-2}{2}\int_Q q\mid u\mid^2 dxdt+\int_Q q m_k u\frac{\partial u}{\partial x_k}dxdt \\
&+\int_{\Sigma_1} a_1\Delta u_1 m_k\frac{\partial^2}{\partial\nu\partial x_k}(u_2-u_1)d\Sigma \\
&+\frac{1}{2}\int_{\Sigma_1} m_k\nu_k(a_1\mid\Delta u_1\mid^2-a_2\mid\Delta u_2\mid^2)d\Sigma.
\end{aligned}
\tag{4.10}
$$

We now estimate the right hand of (4.10). First, by Cauchy-Schwarz's inequality we have

$$
\begin{aligned}
&\Big|\Big(u'(t), m_k\frac{\partial u(t)}{\partial x_k}+\frac{n-2}{2}u(t)\Big)\Big| \\
\le&\frac{R_*\mu_1}{2\sqrt{a_2}}\int_\Omega\mid u'(t)\mid^2 dx \\
&+\frac{a_2}{2R_*\sqrt{a_2}\mu_1}\int_\Omega\Big|m_k\frac{\partial u(t)}{\partial x_k}+\frac{n-1}{2}u(t)\Big|^2dx.
\end{aligned}
\tag{4.11}
$$

Moreover,

$$
\begin{aligned}
&\int_\Omega\Big|m_k\frac{\partial u(t)}{\partial x_k}+\frac{n-2}{2}u(t)\Big|^2dx \\
=&\int_\Omega\Big|m_k\frac{\partial u}{\partial x_k}\Big|^2dx+\frac{(n-2)^2}{4}\int_\Omega\mid u(t)\mid^2 dx \\
&+(n-2)\Big(m_k\frac{\partial u}{\partial x_k}, u(t)\Big).
\end{aligned}
\tag{4.12}
$$

Since

$$
\begin{aligned}
\Big(m_k\frac{\partial u}{\partial x_k}, u(t)\Big)=&\frac{1}{2}\int_\Omega m_k\frac{\partial}{\partial x_k}(\mid u(t)\mid^2)dx \\
=&\frac{1}{2}\int_{\Gamma_1} m_k\nu_k\mid u_1(t)\mid^2 d\Gamma-\frac{n}{2}\int_{\Omega_1}\mid u_1(t)\mid^2 dx \\
&-\frac{1}{2}\int_{\Gamma_1} m_k\nu_k\mid u_2(t)\mid^2 d\Gamma-\frac{n}{2}\int_{\Omega_2}\mid u_2(t)\mid^2 dx \\
=&-\frac{n}{2}\int_\Omega\mid u(t)\mid^2 dx,
\end{aligned}
\tag{4.13}
$$

then by (1.7) and (1.9) we deduce

$$
\begin{aligned}
&\int_\Omega\Big|m_k\frac{\partial u(t)}{\partial x_k}+\frac{n-2}{2}u(t)\Big|^2dx \\
=&\int_\Omega\Big|m_k\frac{\partial u(t)}{\partial x_k}\Big|^2dx+\frac{2^2-n^2}{4}\int_\Omega\mid u(t)\mid^2 dx \\
\le&R^2(x^0)\int_\Omega\mid\nabla u(t)\mid^2 dx+\frac{2^2-n^2}{4}\int_\Omega\mid u(t)\mid^2 dx \\
\le&R_*^2\mu_1^2\int_\Omega\mid\Delta u(t)\mid^2 dx.
\end{aligned}
\tag{4.14}
$$

Thus, (4.11) becomes

$$\Big|\Big(u'(t), m_k\frac{\partial u(t)}{\partial x_k}+\frac{n-2}{2}u(t)\Big)\Big| \le \frac{R_*\mu_1}{\sqrt{a_2}}E(u,t), \quad \forall t\in[0,T]. \tag{4.15}$$

We then estimate the last two terms of (4.10). Since

$$\frac{\partial}{\partial x_k}\Big(\frac{\partial u_1}{\partial \nu}\Big) = \nu_k\frac{\partial^2 u_1}{\partial \nu^2}+\sigma_k\Big(\frac{\partial u_1}{\partial \nu}\Big) \text{ on } \quad \Sigma_1,$$

$$\frac{\partial}{\partial x_k}\Big(\frac{\partial u_2}{\partial \nu}\Big) = \nu_k\frac{\partial^2 u_2}{\partial \nu^2}+\sigma_k\Big(\frac{\partial u_2}{\partial \nu}\Big) \text{ on } \quad \Sigma_1,$$

$$\sigma_k\Big(\frac{\partial u_1}{\partial \nu}\Big)=\sigma_k\Big(\frac{\partial u_2}{\partial \nu}\Big), \quad \frac{\partial^2}{\partial x_k^2}(u_2-u_1)=\nu_k^2\frac{\partial^2}{\partial \nu^2}(u_2-u_1) \quad \text{on } \Sigma_1$$

where σ_k denote the first order tangential differential operators on Σ_1 (see [10], Chapter 3, p.137), we deduce

$$\begin{aligned}
&\int_{\Sigma_1} a_1\Delta u_1 m_k\frac{\partial^2}{\partial\nu\partial x_k}(u_2-u_1)d\Sigma \\
&\quad+\frac{1}{2}\int_{\Sigma_1} m_k\nu_k(a_1\mid\Delta u_1\mid^2-a_2\mid\Delta u_2\mid^2)d\Sigma \\
&=\int_{\Sigma_1} a_1\Delta u_1 m\cdot\nu\Big(\frac{\partial^2 u_2}{\partial\nu^2}-\frac{\partial^2 u_1}{\partial\nu^2}\Big)d\Sigma \\
&\quad+\frac{1}{2}\int_{\Sigma_1} m_k\nu_k(a_1\mid\Delta u_1\mid^2-a_2\mid\Delta u_2\mid^2)d\Sigma \\
&=\int_{\Sigma_1} a_1\Delta u_1 m\cdot\nu\Big(\Delta u_2-\Delta u_1\Big)d\Sigma \\
&\quad+\frac{1}{2}\int_{\Sigma_1} m_k\nu_k(a_1\mid\Delta u_1\mid^2-a_2\mid\Delta u_2\mid^2)d\Sigma \\
&=\frac{a_1(a_1-a_2)}{2a_2}\int_{\Sigma_1} m\cdot\nu\mid\Delta u_1\mid^2 d\Sigma \\
&\ge 0,
\end{aligned} \tag{4.16}$$

since $a_1\ge a_2$ and $m\cdot\nu\ge 0$ on Γ_1. It therefore follows from (4.2), (4.10), (4.15) and (4.16) that

$$\begin{aligned}
&\frac{1}{2}\int_{\Sigma} a_2 m_k\nu_k\mid\Delta u_2\mid^2 d\Sigma \\
&\ge 2\int_0^T E(u,t)dt-\frac{R_*\mu_1}{\sqrt{a_2}}[E(u,0)+E(u,T)] \\
&\quad-\frac{q_0|n-2|}{\lambda_1 a_2}\int_0^T E(u,t)dt-\frac{q_0R_*}{\lambda_1^{3/4}a_2}\int_0^T E(u,t)dt \\
&=(2-q_0|n-2|\lambda_1^{-1}a_2^{-1}-q_0R_*\lambda_1^{-3/4}a_2^{-1})\int_0^T E(u,t)dt \\
&\quad-2R_*\mu_1a_2^{-1/2}E(u,0)+R_*\mu_1a_2^{-1/2}\int_0^T\int_\Omega quu'dxdt \\
&\ge[2-q_0|n-2|\lambda_1^{-1}a_2^{-1}-q_0R_*\lambda_1^{-1/2}a_2^{-1}(\mu_1+\lambda_1^{-1/4})]\int_0^T E(u,t)dt \\
&\quad-2R_*\mu_1a_2^{-1/2}E(u,0).
\end{aligned} \tag{4.17}$$

Moreover, by (4.2) we have

$$\begin{aligned}\int_0^T E(u,t)dt &= TE(u,0) - \int_0^T \int_0^t \int_\Omega qu(x,s)u'(x,s)dxdsdt \\ &\geq TE(u,0) - q_0 T\lambda_1^{-1/2} a_2^{-1/2} \int_0^T E(u,t)dt,\end{aligned}$$

which implies

$$\int_0^T E(u,t)dt \geq \frac{T}{1+q_0 T\lambda_1^{-1/2} a_2^{-1/2}} E(u,0). \tag{4.18}$$

Combining (4.5), (4.17) and (4.18) gives (4.6). □

In order to relax the restrictions on T and $q(x,t)$, we need the following unique theorems.

Theorem 4.2. *Suppose there exists $x^0 \in \Omega_1$ such that $m(x)\cdot\nu(x) \geq 0$ on Γ_1, where ν directs towards the exterior of Ω_1. Assume $a_2 \leq a_1$. Let $q \in W^{1,\infty}(\Omega)$ be such that $m(x)\cdot\nabla q \leq 0$ on Ω. If*

$$\begin{cases} a(x)\Delta^2 u + q(x)u = 0 & \text{in } \Omega, \\ u(0) = u^0, \quad u'(0) = u^1 & \text{in } \Omega, \\ u_2 = \dfrac{\partial u_2}{\partial \nu} = 0, & \text{on } \Gamma, \\ u_1 = u_2, \quad \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} & \text{on } \Gamma_1, \\ a_1\Delta u_1 = a_2 \Delta u_2, \quad a_1\dfrac{\partial \Delta u_1}{\partial \nu} = a_2 \dfrac{\partial \Delta u_2}{\partial \nu} & \text{on } \Gamma_1, \\ \Delta u_2 = 0 & \text{on } \Gamma(x^0), \end{cases} \tag{4.19}$$

then

$$u = 0 \qquad \text{in } \Omega. \tag{4.20}$$

Proof. Multiplying (4.19) by $m_k \dfrac{\partial u}{\partial x_k}$ and integrating over Ω, we obtain

$$\int_\Omega a(x) m_k \frac{\partial u}{\partial x_k} \Delta^2 u dx + \int_\Omega q m_k \frac{\partial u}{\partial x_k} u dx = 0. \tag{4.21}$$

It follows from (2.6) that

$$\begin{aligned}\int_\Omega a(x) m_k \frac{\partial u}{\partial x_k}\Delta^2 u dx &= -\frac{1}{2}\int_\Gamma a_2 m_k \nu_k |\Delta u_2|^2 d\Gamma + (2-\frac{n}{2})\int_\Omega a(x)|\Delta u|^2 dx \\ &+ \int_{\Gamma_1} a_1 \Delta u_1 m_k \frac{\partial^2}{\partial\nu\partial x_k}(u_2 - u_1) d\Gamma \\ &+ \frac{1}{2}\int_{\Gamma_1} m_k\nu_k (a_1 \mid \Delta u_1 \mid^2 - a_2 \mid \Delta u_2 \mid^2) d\Gamma.\end{aligned} \tag{4.22}$$

Moreover, we have

$$\int_\Omega q m_k \frac{\partial u}{\partial x_k} u dx = -\frac{n}{2}\int_\Omega q|u|^2 dx - \frac{1}{2}\int_\Omega m_k \frac{\partial q}{\partial x_k}|u|^2 u dx. \tag{4.23}$$

On the other hand, Multiplying (4.19) by u and integrating over Ω, we obtain

$$\int_\Omega (a(x)|\Delta u|^2 + q|u|^2) dx = 0. \tag{4.24}$$

We then deduce from (4.21)-(4.24) that

$$
\begin{aligned}
0 = &-\frac{1}{2}\int_{\Gamma_*(x^0)} a_2 m_k \nu_k |\Delta u_2|^2 d\Gamma + 2\int_\Omega a(x)|\Delta u|^2 dx - \frac{1}{2}\int_\Omega m_k \frac{\partial q}{\partial x_k}|u|^2 u dx \\
&+ \int_{\Gamma_1} a_1 \Delta u_1 m_k \frac{\partial^2}{\partial\nu\partial x_k}(u_2 - u_1) d\Gamma \\
&+ \frac{1}{2}\int_{\Gamma_1} m_k \nu_k (a_1 \mid \Delta u_1 \mid^2 - a_2 \mid \Delta u_2 \mid^2) d\Gamma.
\end{aligned}
$$

which, by (4.16), implies (4.20) since $m\cdot\nu \le 0$ on $\Gamma_*(x^0)$ and $m(x)\cdot\nabla q \le 0$ on Ω. □

If $-q$ is a constant and not an eigenvalue of $a(x)\Delta^2$, then the condition "$\Delta u_2 = 0$ on $\Gamma(x^0)$" is not required. In addition, there are functions q which satisfy the condition of Theorem 4.2, for example, $q \equiv c$ (a constant) and $q = -|x-x^0|^2$.

The following uniqueness theorem is an extension of a theorem of Zuazua (see [10], Appendix 1) to the case of transmission with lower-order terms.

Theorem 4.3. *Suppose the boundary Γ of Ω is of class C^3. Suppose there exists $x^0 \in \Omega_1$ such that $m(x)\cdot\nu(x) \ge 0$ on Γ_1, where ν directs towards the exterior of Ω_1. Assume $a_2 \le a_1$ and $T > 0$. Let either $q \in L^\infty(0,T;W^{1,\infty}(\Omega))$ be such that $m(x)\cdot\nabla q(x,t_0) \le 0$ on Ω for some $t_0 \in [0,T]$ or $q \in L^\infty(Q)$ be such that*

$$
q_0 < \frac{2}{R_*\lambda_1^{-1/2}a_2^{-1}(3\mu_1 + \lambda_1^{-1/4}) + |n-2|\lambda_1^{-1}a_2^{-1}}. \tag{4.25}
$$

If

$$
u \in X = C([0,T]; H_0^2(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \tag{4.26}
$$

is a solution of

$$
\begin{cases}
u'' + a(x)\Delta^2 u + qu = 0 & \text{in } Q, \\
u_2 = \dfrac{\partial u_2}{\partial\nu} = 0 & \text{on } \Sigma, \\
u_1 = u_2, \quad \dfrac{\partial u_1}{\partial\nu} = \dfrac{\partial u_2}{\partial\nu} & \text{on } \Sigma_1, \\
a_1\Delta u_1 = a_2\Delta u_2, \quad a_1\dfrac{\partial\Delta u_1}{\partial\nu} = a_2\dfrac{\partial\Delta u_2}{\partial\nu} & \text{on } \Sigma_1,
\end{cases} \tag{4.27}
$$

such that

$$
\Delta u_2 = 0 \qquad \text{on } \Sigma(x^0), \tag{4.28}
$$

then

$$
u = 0 \qquad \text{in } Q.
$$

Proof. Set

$$
Y = \{u \in X \;:\; u \text{ satisfies } (4.27),\ (4.28)\}
$$

with the norm induced by X. It suffices to prove $Y = \{0\}$.

Because by (4.2) we have

$$
\begin{aligned}
\int_0^T E(u,t)dt &= TE(u,0) - \int_0^T\int_0^t\int_\Omega quu' dx ds dt \\
&= TE(u,0) - \frac{1}{2}\int_0^T\int_\Omega q(u^2(x,t) - u^2(x,0)) dx dt,
\end{aligned}
$$

we deduce from (1.12) and (4.10) that

$$
\begin{aligned}
E(u,0) &\le c_1 \parallel \Delta u_2 \parallel^2_{L^2(\Sigma(x^0))} + c_2 \parallel u \parallel^2_{L^\infty(0,T;H^1_0(\Omega))} \\
&\quad + \left| \left(u'(t), m_k \frac{\partial u(t)}{\partial x_k} + \frac{n-2}{2} u(t) \right) \Big|_0^T \right| \\
&\le c_1 \parallel \Delta u_2 \parallel^2_{L^2(\Sigma(x^0))} + c(\varepsilon) \parallel u \parallel^2_{L^\infty(0,T;H^1_0(\Omega))} + \varepsilon E(u,0),
\end{aligned}
$$

which implies

$$E(u,0) \le c[\parallel \Delta u_2 \parallel^2_{L^2(\Sigma(x^0))} + \parallel u \parallel^2_{L^\infty(0,T;H^1_0(\Omega))}], \tag{4.29}$$

for $u \in X$ satisfying (4.27).

On the other hand, we show there exists a constant $c > 0$ such that

$$\parallel u \parallel^2_{L^\infty(0,T;H^1_0(\Omega))} \le c[\parallel \Delta u_2 \parallel^2_{L^2(\Sigma(x^0))} + \parallel u \parallel^2_{L^\infty(0,T;L^2(\Omega))}] \tag{4.30}$$

for $u \in X$ satisfying (4.27). In fact, if (4.30) is not true, there exists a sequence $\{u_n\}$ of solutions of (4.27) with (4.26) such that

$$\parallel \Delta u_{n2} \parallel^2_{L^2(\Sigma(x^0))} + \parallel u_n \parallel^2_{L^\infty(0,T;L^2(\Omega))} \to 0 \ (n \to \infty) \tag{4.31}$$

and

$$\parallel u_n \parallel_{L^\infty(0,T;H^1_0(\Omega))} = 1. \tag{4.32}$$

It therefore follows from (4.29) and (1.12) that $\{u_n\}$ is bounded in X, and then relatively compact in $L^\infty(0,T;H^1_0(\Omega))$ because the injection: $X \to L^\infty(0,T;H^1_0(\Omega))$ is compact due to Simon's results [13]. By extracting a subsequence, we may as well assume $\{u_n\}$ converges strongly to u in $L^\infty(0,T;H^1_0(\Omega))$. Thus by (4.32) we obtain

$$\parallel u \parallel_{L^\infty(0,T;H^1_0(\Omega))} = 1. \tag{4.33}$$

However, (4.31) implies

$$u = 0 \qquad \text{in } Q.$$

This is in contradiction with (4.33).

By (4.29) and (4.30) we have

$$E(u,0) \le c[\parallel \Delta u_2 \parallel^2_{L^2(\Sigma(x^0))} + \parallel u \parallel^2_{L^\infty(0,T;L^2(\Omega))}] \tag{4.34}$$

for $u \in X$ satisfying (4.27). By the argument of density, (4.34) still holds for $u \in L^\infty(0,T;L^2(\Omega))$ satisfying (4.27).

We observe if $u \in Y$ then $v = u'$ satisfies (4.27) and (4.28), and $v \in L^\infty(0,T;L^2(\Omega))$. Hence by (4.34) we deduce that

$$(v(0), v'(0)) \in H^2_0(\Omega) \times L^2(\Omega),$$

and then by (1.12) we deduce $v \in X$. It therefore follows that

$$u \to u' \text{ is a continuous operator from } Y \text{ to } Y, \tag{4.35}$$

since by (4.34) $\dfrac{d}{dt} : Y \to Y$ maps a bounded subset of Y into a bounded subset of Y. Moreover, the injection $\{u \in Y \ : \ u' \in Y\} \to Y$ is compact. Thus we deduce that the dimension of Y is finite.

Suppose $Y \ne \{0\}$. Then by complexifying Y, it follows from (4.35) that there exists $\lambda \in \mathbf{C}$ and $u \in Y - \{0\}$ such that

$$u' = \lambda u.$$

This implies

$$u(x,t)=e^{\lambda t}u(x,0). \tag{4.36}$$

Since

$$\Delta u_2(x,0)=0 \qquad \text{on } \Gamma(x^0), \tag{4.37}$$

it follows from (4.36) that

$$\Delta u_2(x,t)=0 \qquad \text{on } \Gamma(x^0)\times(-\infty,+\infty). \tag{4.38}$$

On the other hand, if $q\in L^\infty(Q)$ such that (4.25) holds, then we can find $T_0>R_*\mu_1 a_2^{-1/2}$ such that

$$q_0<\frac{2(T_0-R_*\mu_1 a_2^{-1/2})}{T_0[R_*\lambda_1^{-1/2}a_2^{-1}(3\mu_1+\lambda_1^{-1/4})+|n-2|\lambda_1^{-1}a_2^{-1}]}.$$

Then by Lemma 4.1 we deduce

$$u=0 \qquad \text{in } Q. \tag{4.39}$$

If $q\in L^\infty(0,T;W^{1,\infty}(\Omega))$ satisfying $m(x)\cdot\nabla q(x,t_0)\le 0$ on Ω for some $t_0\in[0,T]$, then we substitute (4.36) into (4.27) and we obtain

$$\begin{cases} a(x)\Delta^2 u(x,0)+(q(x,t_0)+\lambda^2)u(x,0)=0 & \text{in } \Omega,\\ u_2(x,0)=\dfrac{\partial u_2(x,0)}{\partial\nu}=0, & \text{on } \Gamma,\\ u_1(x,0)=u_2(x,0),\quad \dfrac{\partial u_1(x,0)}{\partial\nu}=\dfrac{\partial u_2(x,0)}{\partial\nu} & \text{on } \Gamma_1,\\ a_1\Delta u_1(x,0)=a_2\Delta u_2(x,0),\quad a_1\dfrac{\partial\Delta u_1(x,0)}{\partial\nu}=a_2\dfrac{\partial\Delta u_2(x,0)}{\partial\nu} & \text{on } \Gamma_1,\\ \Delta u_2(x,0)=0 & \text{on } \Gamma(x^0). \end{cases}$$

By Lemma 4.2, we also have (4.39). This is in contradiction with $u\in Y-\{0\}$. □

Using Theorem 4.3, we prove

Lemma 4.4. (The observability inequality) *Suppose the boundary Γ of Ω is of class C^3 and the boundary Γ_1 of Ω_1 is of class C^4. Suppose there exists $x^0\in\Omega_1$ such that $m(x)\cdot\nu(x)\ge 0$ on Γ_1, where ν directs towards the exterior of Ω_1. Assume $a_2\le a_1$ and $T>0$. Let $q\in L^\infty(0,T;W^{1,\infty}(\Omega))$ be such that $m(x)\cdot\nabla q(x,t_0)\le 0$ on Ω for some $t_0\in[0,T]$ or $q\in L^\infty(Q)$ be such that (4.25) holds. Then there is $c>0$ such that for all weak solutions u of (1.1) with $f=0$ we have*

$$E(0)\le c\int_{\Sigma(x^0)}|\Delta u_2|^2\,d\Sigma. \tag{4.40}$$

Proof. By (4.29), there exists a constant $c>0$ such that

$$E(u,0)\le c[\|\Delta u_2\|^2_{L^2(\Sigma(x^0))}+\|u\|^2_{L^\infty(0,T;H_0^1(\Omega))}] \tag{4.41}$$

for solutions u of (1.1) with $f=0$.

Furthermore, we show there exists a constant $c>0$ such that

$$\|u\|_{L^\infty(0,T;H_0^1(\Omega))}\le c\|\Delta u_2\|_{L^2(\Sigma(x^0))} \tag{4.42}$$

for solutions u of (1.1) with $f=0$. In fact, if (4.42) is not true, there exists a sequence $\{u_n\}$ of solutions of (1.1) with $f=0$ such that

$$\|\Delta u_{n2}\|_{L^2(\Sigma(x^0))}\to 0\ (n\to\infty) \tag{4.43}$$

and
$$\| u_n \|_{L^\infty(0,T;H_0^1(\Omega))} = 1. \tag{4.44}$$

It therefore follows from (4.41) and (1.12) that $\{u_n\}$ is bounded in $C([0,T];\ H_0^2(\Omega)) \cap C^1([0,T];L^2(\Omega))$, and then relatively compact in $L^\infty(0,T;H_0^1(\Omega))$ because the injection

$$C([0,T];H_0^2(\Omega)) \cap C^1([0,T];L^2(\Omega)) \to L^\infty(0,T;H_0^1(\Omega))$$

is compact due to Simon's results [13]. By extracting a subsequence, we may as well assume $\{u_n\}$ converges strongly to u in $L^\infty(0,T;H_0^1(\Omega))$. Thus by (4.44) we obtain

$$\| u \|_{L^\infty(0,T;H_0^1(\Omega))} = 1. \tag{4.45}$$

In addition, $\{u_n\}$ and $\{u_n'\}$ converge to u star-weakly in $L^\infty(0,T;H_0^2(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$, respectively. Thus u is a solution of (1.1) with

$$C([0,T];H_0^2(\Omega)) \cap C^1([0,T];L^2(\Omega)).$$

By (4.41), we have

$$E(u_n - u_m, 0) \le c[\| \Delta u_{n2} - \Delta u_{m2} \|^2_{L^2(\Sigma(x^0))} + \| u_n - u_m \|^2_{L^\infty(0,T;H_0^1(\Omega))}],$$

which gives
$$E(u_n - u, 0) \le c[\| \Delta u_{n2} \|^2_{L^2(\Sigma(x^0))} + \| u_n - u \|^2_{L^\infty(0,T;H_0^1(\Omega))}]. \tag{4.46}$$

By (2.7), we have
$$\| \Delta u_{n2} - \Delta u_2 \|^2_{L^2(\Sigma(x^0))} \le cE(u_n - u, 0). \tag{4.47}$$

It therefore follows from (4.43), (4.46) and (4.47) that

$$\Delta u_2 = 0 \qquad \text{on } \Sigma(x^0).$$

By Theorem 4.3 we deduce
$$u = 0 \qquad \text{in } Q. \tag{4.48}$$

This is in contradiction with (4.45).

Finally, (4.40) follows from (4.41) and (4.42). □

Lemma 4.5. (The observability inequality) *In addition to all assumptions of Lemma 4.4, suppose the boundary Γ of Ω is of class C^4 and q is independent of t with $q \in W^{2,\infty}(\Omega)$. Then there is $c > 0$ such that for all strong solutions u of (1.1) with $f = 0$ we have*

$$\|u^0\|^2_{H^4(\Omega,\Gamma_1)} + \| u^1 \|^2_{2,\Omega} \le c \int_{\Sigma(x^0)} (| \Delta u_2' |^2 + | \Delta u_2 |^2) d\Sigma. \tag{4.49}$$

Proof. Set $w = u'$. Then w satisfies

$$\begin{cases} w'' + a(x)\Delta^2 w + qw = 0 & \text{in } Q, \\ w(0) = u^1, \quad w'(0) = -a(x)\Delta^2 u^0 - qu^0 & \text{in } \Omega, \\ w_2 = \dfrac{\partial w_2}{\partial \nu} = 0 & \text{on } \Sigma, \\ w_1 = w_2, \quad \dfrac{\partial w_1}{\partial \nu} = \dfrac{\partial w_2}{\partial \nu} & \text{on } \Sigma_1, \\ a_1 \Delta w_1 = a_2 \Delta w_2, \quad a_1 \dfrac{\partial \Delta w_1}{\partial \nu} = a_2 \dfrac{\partial \Delta w_2}{\partial \nu} & \text{on } \Sigma_1. \end{cases} \tag{4.50}$$

It therefore follows from Lemma 4.4 that

$$\begin{aligned}\int_{\Sigma(x^0)} | \Delta u_2' |^2 \, d\Sigma = \int_{\Sigma(x^0)} | \Delta w_2 |^2 \, d\Sigma \\ \geq c[\|a\Delta^2 u^0\|_0^2 - q_0\|u^0\|_0^2 + \| u^1 \|_{2,\Omega}^2],\end{aligned} \tag{4.51}$$

which, by Lemma 4.4, gives (4.49). □

If $q=0$, then (4.49) becomes

$$\|u^0\|^2_{H^4(\Omega,\Gamma_1)} + \| u^1 \|^2_{2,\Omega} \leq c \int_{\Sigma(x^0)} | \Delta u_2' |^2 \, d\Sigma.$$

Lemma 4.6. (The observability inequality) *In addition to all assumptions of Lemma 4.4, suppose the boundary Γ of Ω is of class C^4 and q is independent of t with $q \in W^{2,\infty}(\Omega)$. Let q be such that $a(x)\Delta^2 + q$ is an isomorphism from $H_0^2(\Omega)$ onto $H^{-2}(\Omega)$. Then there is $c>0$ such that for all ultraweak solutions u of (1.1) with $f=0$ we have*

$$\| \Delta u_2 \|_{H^{-1}(0,T;L^2(\Gamma(x^0)))} \geq c\Big[\| u^1 \|_{-2,\Omega} + \| u^0 \|_{0,\Omega}\Big]. \tag{4.52}$$

Proof. Let e be the solution of

$$\begin{cases} a(x)\Delta^2 e + qe = -u^1 & \text{in } \Omega, \\ e_2 = \dfrac{\partial e_2}{\partial \nu} = 0 & \text{on } \Gamma, \\ e_1 = e_2, \quad \dfrac{\partial e_1}{\partial \nu} = \dfrac{\partial e_2}{\partial \nu} & \text{on } \Gamma_1, \\ a_1 \Delta e_1 = a_2 \Delta e_2, \quad a_1 \dfrac{\partial \Delta e_1}{\partial \nu} = a_2 \dfrac{\partial \Delta e_2}{\partial \nu} & \text{on } \Gamma_1, \end{cases}$$

and set

$$w = \int_0^t u(s)ds + e,$$

where u is the solution of (1.1) with $f=0$. It is easily verified that

$$\begin{cases} w'' + a(x)\Delta^2 w + qw = 0 & \text{in } Q, \\ w(0) = e, \quad w'(0) = u^0 & \text{in } \Omega, \\ w_2 = \dfrac{\partial w_2}{\partial \nu} = 0 & \text{on } \Sigma, \\ w_1 = w_2, \quad \dfrac{\partial w_1}{\partial \nu} = \dfrac{\partial w_2}{\partial \nu} & \text{on } \Sigma_1, \\ a_1 \Delta w_1 = a_2 \Delta w_2, \quad a_1 \dfrac{\partial \Delta w_1}{\partial \nu} = a_2 \dfrac{\partial \Delta w_2}{\partial \nu} & \text{on } \Sigma_1. \end{cases}$$

Since $(e,u^0) \in H_0^2(\Omega) \times L^2(\Omega)$, It follows from Lemma 4.4 that there exists a constant $c>0$

$$\begin{aligned}\int_{\Sigma} | \Delta w_2 |^2 \, d\Sigma \geq c\Big[\| e \|^2_{2,\Omega} + \| u^0 \|^2_{0,\Omega}\Big] \\ \geq c\Big[\| u^1 \|^2_{-2,\Omega} + \| u^0 \|^2_{0,\Omega}\Big].\end{aligned} \tag{4.53}$$

Here we have used the fact that $a(x)\Delta^2 + q$ is an isomorphism from $H_0^2(\Omega)$ onto $H^{-2}(\Omega)$. Because $\dfrac{\partial}{\partial t}$ is an isomorphism from $L^2(\Sigma)$ onto $H^{-1}(0,T;\ L^2(\Gamma))$, we conclude

$$\Delta u_2 = \frac{\partial}{\partial t}(\Delta w_2) \in H^{-1}(0,T;L^2(\Gamma)), \tag{4.54}$$

and

$$\| \Delta u_2 \|_{H^{-1}(0,T;L^2(\Gamma))} \geq c\Big[\| u^1 \|_{-2,\Omega} + \| u^0 \|_{0,\Omega}\Big]. \qquad \square$$

§5. Exact Controllability

We are now ready to present main theorems of this paper.

Theorem 5.1. *Suppose the boundary Γ of Ω is of class C^3. Suppose there exists $x^0 \in \Omega_1$ such that $m(x)\cdot\nu(x) \geq 0$ on Γ_1, where ν directs towards the exterior of Ω_1. Assume $a_2 \leq a_1$ and $T > 0$. Let either $q \in L^\infty(0,T;W^{1,\infty}(\Omega))$ be such that $m(x)\cdot\nabla q(x,t_0) \leq 0$ on Ω for some $t_0 \in [0,T]$ or $q \in L^\infty(Q)$ be such that*

$$q_0 < \frac{2}{R_*\lambda_1^{-1/2}a_2^{-1}(3\mu_1 + \lambda_1^{-1/4}) + |n-2|\lambda_1^{-1}a_2^{-1}}. \tag{5.1}$$

Then for all initial states $(y^0, y^1) \in L^2(\Omega)\times H^{-2}(\Omega)$, there exists a control

$$\phi \in L^2(\Sigma) \text{ with } \phi = 0 \text{ on } \Sigma_*(x^0) \tag{5.2}$$

driving system (0.1) to rest.

Proof. We apply HUM. We first consider the problem:

$$\begin{cases} u'' + a(x)\Delta^2 u + qu = 0 & \text{in } Q,\\ u(0) = u^0, \quad u'(0) = u^1 & \text{in } \Omega,\\ u_2 = \dfrac{\partial u_2}{\partial \nu} = 0 & \text{on } \Sigma,\\ u_1 = u_2, \quad \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} & \text{on } \Sigma_1,\\ a_1\Delta u_1 = a_2\Delta u_2, \quad a_1\dfrac{\partial \Delta u_1}{\partial \nu} = a_2\dfrac{\partial \Delta u_2}{\partial \nu} & \text{on } \Sigma_1. \end{cases} \tag{5.3}$$

For any $(u^0, u^1) \in H_0^2(\Omega)\times L^2(\Omega)$, problem (5.3) has a unique solution u with

$$\Delta u_2 \in L^2(\Sigma), \tag{5.4}$$

because of Theorem 1.4 and Lemma 2.2.

Using the solution u of (5.3), we then consider the backward problem:

$$\begin{cases} v'' + a(x)\Delta^2 v + qv = 0 & \text{in } Q,\\ v(T) = v'(T) = 0 & \text{in } \Omega,\\ v_2 = 0, & \text{on } \Sigma,\\ \dfrac{\partial v_2}{\partial \nu} = \begin{cases} \Delta u_2 & \text{on } \Sigma(x^0),\\ 0 & \text{on } \Sigma_*(x^0), \end{cases} & \\ v_1 = v_2, \quad \dfrac{\partial v_1}{\partial \nu} = \dfrac{\partial v_2}{\partial \nu} & \text{on } \Sigma_1,\\ a_1\Delta v_1 = a_2\Delta v_2, \quad a_1\dfrac{\partial \Delta v_1}{\partial \nu} = a_2\dfrac{\partial \Delta v_2}{\partial \nu} & \text{on } \Sigma_1. \end{cases} \tag{5.5}$$

It follows from Theorem 3.2 that problem (5.5) has a unique ultraweak solution v with

$$v \in C([0,T]; L^2(\Omega)) \times C^1([0,T]; H^{-2}(\Omega)). \tag{5.6}$$

We then define a linear operator Λ by

$$\Lambda(u^0, u^1) = (v'(0), -v(0)). \tag{5.7}$$

Multiplying the first equation of (5.3) by v and integrating over Q, we find

$$\begin{aligned} \langle \Lambda(u^0,u^1),(u^0,u^1)\rangle &= \langle v'(0), u^0\rangle - \int_\Omega v(0)u^1 \, dx \\ &= \int_{\Sigma(x^0)} a_2 \mid \Delta u_2 \mid^2 d\Sigma. \end{aligned} \tag{5.8}$$

It therefore follows from Lemma 2.2, Lemma 4.4, and the Lax-Milgram Theorem that Λ is an isomorphism from $H_0^2(\Omega) \times L^2(\Omega)$ onto $H^{-2}(\Omega) \times L^2(\Omega)$. This means that for all $(y^1, -y^0) \in H^{-2}(\Omega) \times L^2(\Omega)$, the equation

$$\Lambda(u^0, u^1) = (y^1, -y^0) \tag{5.9}$$

has a unique solution (u^0, u^1). With this initial condition we solve Problem (5.3), and then solve Problem (5.5). Then we have found a control

$$\phi = \begin{cases} \Delta u_2 & \text{on } \Sigma(x^0), \\ 0 & \text{on } \Sigma_*(x^0), \end{cases} \tag{5.10}$$

such that

$$y(x,t;\phi) = v(x,t;\phi) \tag{5.11}$$

is the solution of (0.1) satisfying

$$y(x,T;\phi) = y'(x,T;\phi) = 0 \qquad \text{in } \Omega. \tag{5.12}$$

This completes the proof. □

Based on Lemma 2.3 and Lemma 4.5, we obtain

Theorem 5.2. *Suppsoe all assumptions of Lemma 4.5 holds. Then for all initial states*

$$(y^0, y^1) \in H^{-2}(\Omega) \times (H^4(\Omega, \Gamma_1))', \tag{5.13}$$

there exists a control

$$\phi \in (H^1(0,T; L^2(\Gamma(x^0))))' \tag{5.14}$$

driving system (0.1) to rest.

Proof. The proof is the same as the one of Theorem 5.1 except that $H_0^2(\Omega) \times L^2(\Omega)$ is replaced by $H^4(\Omega, \Gamma_1) \times H_0^2(\Omega)$ and Δu_2 in (5.5) by $-\frac{\partial}{\partial t}(\Delta u_2') + \Delta u_2$. At this time, we get a control

$$\phi = \begin{cases} -\dfrac{\partial}{\partial t}(\Delta u_2') + \Delta u_2 & \text{on } \Sigma(x^0), \\ 0 & \text{on } \Sigma_*(x^0). \end{cases} \tag{5.15}$$

Based on Lemma 3.8 and Lemma 4.6, we obtain

Theorem 5.3. *Suppsoe all assumptions of Lemma 4.6 holds. Then for all initial states*

$$(y^0, y^1) \in H_0^2(\Omega) \times L^2(\Omega), \tag{5.16}$$

there exists a control

$$\phi \in H_0^1(0,T;L^2(\Gamma(x^0))) \tag{5.17}$$

driving system (0.1) to rest.

Proof. The proof is the same as the one of Theorem 5.1 except that $H_0^2(\Omega)\times L^2(\Omega)$ is replaced by $L^2(\Omega)\times H^{-2}(\Omega)$ and Δu_2 in (5.5) by ψ, where $\psi \in H_0^1(0,T;L^2(\Gamma(x^0)))$ is such that

$$\langle \Delta u_2, \psi\rangle = \| \Delta u_2 \|^2_{H^{-1}(0,T;L^2(\Gamma(x^0)))} . \tag{5.18}$$

At this time, we get a control

$$\phi = \begin{cases} \psi, & \text{on } \Sigma(x^0), \\ 0, & \text{on } \Sigma_*(x^0). \end{cases}$$

□

Acknowledgement

The first author is grateful for the Overseas Postgraduate Research Scholarship from Australian Government, the University of Wollongong Postgraduate Research Award, and the Department of Mathematics Analysis Research Group Scholarship from the Faculty of Informatics at the University of Wollongong. The authors thank the anonymous referee for bringing the papers [1], [4], [5], [9] to their attention.

References

[1] N. Burq, Contrôle de l'équation des plaques en présence d'obstacles strictement convexes, *Mém. Soc. Math. France (N.S.),* 55 (1993), 1-126.

[2] R. Dautray and J.L. Lions, Mathematical analysis and numerical methods for science and technology, Vol.2, Functional and variational methods, Springer-Verlag, Berlin, 1992.

[3] R. Dautray and J.L. Lions, Mathematical analysis and numerical methods for science and technology, Vol.5, Evolution problems I, Springer-Verlag, Berlin, 1992.

[4] A. Haraux, Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire, *J. Math. Pures Appl.,* 68 (1989), 457-465.

[5] S. Jaffard, Contrôle interne exact des vibrations d'une plaque carrée, *C.R. Acad. Sci. Paris, Série I*, 307 (1988), 759-762.

[6] V. Komornik, Exact controllability and stabilization: The multiplier method, John Wiley & Sons, Masson, Paris, 1994.

[7] I. Lasiecka, and R. Triggiani, Exact controllability of the Euler-Bernoulli equation with controls in the Dirichlet and Neumann boundary conditions: A nonconservative case, *SIAM J. Control Optim.,* 27 (1989), 330-373.

[8] I. Lasiecka and R. Triggiani, Further results on exact controllability of the Euler-Bernoulli equation with controls on the Dirichlet and Neumann boundary conditions, Stabilization of Flexible structures (Montpellier, 1989), Lecture Notes in Control and Information Sciences, 147, Springer-Verlag, 1990, 226-234.

[9] G. Lebeau, Contrôle de l'équation de Schrödinger, *J. Math. Pures Appl.,* 71 (1992), 267-291.

[10] J.L. Lions, Contrôlabilité exacte perturbations et stabilisation de systè- mes distribués, Tome 1, Contrôlabilité exacte, Masson Paris Milan Mexico, 1988.

[11] J.L. Lions, Exact controllability, stabilization and perturbations for distributed systems, *SIAM Rev.,* 30 (1988), 1-68.

[12] J.L. Lions and E. Magenes, Non-homogeneous boundary value problems and applications, Vol. I and II, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[13] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl. (IV)*, CXLVI (1987), 65-96.

[14] E. Zuazua, Exact boundary controllability for the semilinear wave equation, in Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar, Vol X (Paris, 1987-1988) 357-391, Pitman Research Notes in Mathematics Series 220, Longman Science and Technology, Harlow, 1991 (H.O. Fattorini).