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Decay Rates for Dissipative Wave equations *

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Abstract

We derive decay rates for the energy of solutions of dissipative wave equations. The metod of proof combines multiplier techniques and the construction of suitable Lyapunov functionals. Without imposing any growth condition at the origin on the nonlinearity we show that this Lyapunov functional, which is equivalent to the energy of the system, is bounded above by the solution of a differential inequality that tends to zero as time goes to infinity.

Key Words: Decay rate of energy; wave equation; nonlinear dissipation; Lyapunov functional; differential inequality.

AMS subject classification: 35B35, 35L05.

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1 Introduction

Let Ω be a bounded domain in $\mathbf{R}^n$ with suitably smooth boundary $\Gamma = \partial\Omega$ and consider the following wave equation with a nonlinear internal damping

$$\begin{cases} u'' - \Delta u + g(u') = 0 & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{in } \Gamma \times (0, \infty), \\ u(0) = u^0,\ u'(0) = u^1 & \text{in } \Omega. \end{cases} \tag{1.1}$$

By $'$ we denote the derivative with respect to the time variable. Δ denotes the Laplace operator in the space variables. $u(0)$, $u'(0)$ denote the functions $x \to u(x, 0)$, $x \to u'(x, 0)$, respectively. $g(s) \in C(\mathbf{R})$ is a given function.

All along this paper we assume g to be non-decreasing and such that $g(0) = 0$. Under these conditions the nonlinear term $g(u')$ has a dissipative effect on the equation. Indeed, the energy

$$E(t) = \frac{1}{2} \int_\Omega (|u'|^2 + |\nabla u|^2) dx \tag{1.2}$$

of solutions of (1.1) satisfies

$$\frac{dE(t)}{dt} = - \int_\Omega g(u')u' dx. \tag{1.3}$$

It is well known that (see [6, 7, 8, 9, 10, 18, 19]) if g satisfies the following polynomial growth assumption near the origin

$$c|s|^r \le |g(s)| \le C|s|^{1/r}, \quad \forall s \in \mathbf{R} \ \text{ with } \ |s| \le 1, \tag{1.4}$$

where $c > 0$ and $r \ge 1$ are constant, then the energy E decays polynomially as $t \to \infty$. More precisely, for every solution of (1.1) there exists a positive constant C that depends continuously on the initial energy $E(0)$ such that

$$E(t) \le C(E(0)) t^{-2/(r-1)}, \quad \forall t > 0. \tag{1.5}$$

When $r = 1$ the decay rate is exponential. In order for this decay rate to hold some minimal growth conditions on the nonlinearity are also needed at infinity. Namely,

$$c|s| \le |g(s)| \le C|s|^p, \quad \forall s : |s| \ge 1 \tag{1.6}$$

with $p > 1$ such that $(n-2)p \le n + 2$. Note that the decay rate (1.5) depends only on the behavior of the nonlinearity at the origin. However, the constant $C(E(0))$ appearing in (1.5) does depend also on the growth conditions (1.6) (see [4]).

The aim of this paper is to obtain an explicit decay rate for the energy of solutions of (1.1) without any growth assumption on the nonlinear damping term g near the origin. More precisely, given any continuous, non-decreasing function such that $g(0) = 0$ and satisfying the growth conditions at infinity (1.6) we intend to obtain a decay rate for the energy. In this setting (1.5) should just be a particular example of a rather general result valid under the further condition (1.4).

In particular one could ask what the decay rate of solutions is when the nonlinearity g degenerates near the origin faster than any polynomial. For example, if g satisfies the following condition

$$|g(s)| = e^{-1/s^2}, \qquad \text{for } |s| \leq 1. \tag{1.7}$$

This problem was studied by Lasiecka and Tataru [11] in the context of the nonlinear boundary damping, who proved that the decay rate can be described by a dissipative ordinary differential equation without imposing any growth condition to the nonlinearity at the origin. We here give an easy proof of this result in the case of the internally distributed damping that provides a simpler dissipative ordinary differential equation describing the decay rate. We employ the method introduced in [7], [19] and [20] based on the construction of a suitable Lyapunov functional which is equivalent to the energy of the system. We prove that this Lyapunov functional satisfies the desired differential inequality using Young's and Jensen's inequalities. The proof is rather constructive and therefore, given a dissipative function g, this ordinary differential equation can be easily constructed explicitly. This allows to recover the classical polynomial decay rates under the condition (1.4) but also to prove logarithmic decay rates for nonlinearities that degenerate exponentially at the origin as in (1.7).

There is an extensive literature in this topic and our list of references is not exhaustive at all. In addition to the methods we have indicated above the works by M. Nakao [14] and [15] are also worth mentioning. In these works an integral inequality is derived for the energy. The decay rate then holds by solving this integral inequality. This method has been recently greatly extended by P. Martinez [12] and [13] who has obtained results that are close to the one we present here in the case where the damping is localized in a suitable open subset of the domain or on the boundary.

Very recently, J. Vascontenoble [17] has done an important contribution to this field showing that the classical decay estimates for the energy of dissipative wave equations (both when the dissipative term acts in the interior or on the boundary) are optimal.

The rest of the paper is organized as follows. We present our main result and its proof in section 2. Then, in section 3, we give three examples to illustrate how to derive from our general result the usual exponential or polynomial decay rate and also the logarithmic decay rate for the damping that degenerates exponentially at the origin.

2 Main Result and Proof

In what follows, $H^s(\Omega)$ denotes the usual Sobolev space (see [1]) for any $s \in \mathbf{R}$. For $s \geq 0$, $H_0^s(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in $H^s(\Omega)$, where $C_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions on Ω with compact support in Ω. Let X be a Banach space. We denote by $C^k([0,T];X)$ the space of all k times continuously differentiable functions defined on $[0,T]$ with values in X, and write $C([0,T];X)$ for $C^0([0,T];X)$.

We define the energy $E(t)$ of solutions of (1.1) by

$$E(t) = \frac{1}{2}\int_\Omega (|u'|^2 + |\nabla u|^2)dx. \tag{2.1}$$

Let $k > 0$ denote the best constant in Poincaré's inequality

$$\int_\Omega |u|^2 dx \leq k^2 \int_\Omega |\nabla u|^2 dx, \quad \forall\, u \in H_0^1(\Omega). \tag{2.2}$$

Let the constant p satisfy

$$1 \leq p \leq \frac{n+2}{n-2}, \quad \text{if } n > 2, \tag{2.3}$$

$$1 \leq p < \infty, \quad \text{if } n \leq 2. \tag{2.4}$$

Then, by Sobolev's embedding theorem (see [1, p. 97]), $H_0^1(\Omega)$ is embedded into $L^{p+1}(\Omega)$, and consequently, $L^{(p+1)/p}(\Omega)$ is continuously embedded into $H^{-1}(\Omega)$. Let $\alpha = \alpha(p) > 0$ denote the best constant such that

$$\|u\|_{H^{-1}(\Omega)} \leq \alpha\Big(\int_\Omega |u|^{(p+1)/p} dx\Big)^{p/(p+1)}, \quad \forall\, u \in L^{(p+1)/p}(\Omega). \tag{2.5}$$

The following is our main result.

Theorem 2.1 *Assume that $g \in C(\mathbf{R})$ satisfies the following conditions:*

(i) $g(0) = 0$;

(ii) g is increasing on $\mathbf{R}$;

(iii) there are constants $c_1, c_2 > 0$ and $p \geq 1$ satisfying (2.3)-(2.4) such that

$$c_1|s| \leq |g(s)| \leq c_2|s|^p, \quad \textit{for } |s| \geq 1; \tag{2.6}$$

(iv) there exists a strictly increasing positive function $h(s)$ of class C^2 defined on $[0, \infty)$ and a constant $c_3 > 0$ such that

$$c_3 h(|s|) \leq |g(s)| \leq c_4 h^{-1}(|s|), \quad \textit{for } |s| \leq 1, \tag{2.7}$$

where h^{-1} denotes the inverse of h and $c_4 = \max_{|s|\leq 1} |g(s)|$;

(v) there exists an increasing, positive and convex function $\varphi = \varphi(s)$ defined on $[0, \infty)$ and twice differentiable outside $s = 0$ such that $\varphi(|s|^{(p+1)/p}) \leq h(|s|)|s|$ on $[-1, 1]$ and $\varphi''(s)s$ is increasing on $[0, \infty)$.

Then the energy $E(t)$ of solutions of (1.1) with $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfies the following decay rate:

$$E(t) \leq 2V(t), \quad \textit{for } t \geq 0, \tag{2.8}$$

where $V(t)$ is the solution of the following differential equation:

$$\begin{aligned} V'(t) &= -\frac{\epsilon V(t)}{b}\varphi'\Big(\frac{aV(t)}{b}\Big) - \epsilon m_1 \varphi\Big(\frac{aV(t)}{b}\Big) \\ &\quad + m_2 \epsilon \lambda^{p+1} \varphi'\Big(\frac{aV(t)}{c}\Big)\Big(\frac{V(t)}{c}\Big)^{(p+1)/2}. \end{aligned} \tag{2.9}$$

Furthermore, we have

$$\lim_{t\to\infty} E(t) = \lim_{t\to\infty} V(t) = 0. \tag{2.10}$$

The various constants above are given by

$$\lambda = \textit{any positive constant (very small in practice)}, \tag{2.11}$$

$$m_1 = 2\mathrm{mes}(\Omega) + \frac{\alpha p}{p+1} c_4^{(p+1)/p} \mathrm{mes}(\Omega) \lambda^{-(p+1)/p}, \tag{2.12}$$

$$m_2 = \frac{\alpha}{p+1} 2^{(p+1)/2}(1 + c_2^{1/(p+1)}), \tag{2.13}$$

$$
\begin{aligned}
a &= m_1^{-1}, && (2.14)\\
M_1 &= (ak\varphi''(aE(0))E(0) + 2c_1^{-1}\varphi'(aE(0)) + \frac{\alpha p}{p+1}c_2^{1/(p+1)}\lambda^{-(p+1)/p}\varphi'(aE(0)))^{-1}, && (2.15)\\
M_2 &= (ak\varphi''(aE(0))E(0) + \frac{\alpha p}{p+1}c_4^{1/p}\lambda^{-(p+1)/p} + 2c_3^{-1})^{-1}, && (2.16)\\
\epsilon &= \min\{M_1,\ M_2,\ \frac{1}{2k\varphi'(aE(0))}\}, && (2.17)\\
b &= 1 + \epsilon k\varphi'(aE(0)), && (2.18)\\
c &= 1 - \epsilon k\varphi'(aE(0)). && (2.19)
\end{aligned}
$$

Proof. We may assume that $(u^0, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ so that the solutions have the following regularity

$$
u \in C([0,\infty), H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,\infty), H_0^1(\Omega)). \tag{2.20}
$$

The general case can be handled by a density argument.

By a straightforward calculation, we have

$$
E'(t) = -\int_\Omega u'g(u')dx \leq 0. \tag{2.21}
$$

If $E(t_0) = 0$ for some $t_0 \geq 0$, then, by (2.21), we have $E(t) \equiv 0$ for $t \geq t_0$ and then the theorem holds. Therefore, we may assume that $E(t) > 0$ for $t \geq 0$. This assumption ensures that, in the following proof, $\varphi''(aE(t))$ makes sense as we have assumed that $\varphi(s)$ is twice differentiable outside $s = 0$.

Set

$$
V(t) = E(t) + \epsilon\psi(E(t)) \int_\Omega uu'dx, \tag{2.22}
$$

where $\psi(s)$ and $\psi'(s)s$ are positive and increasing functions on $(0, +\infty)$ that will be determined in the proof. Using (2.21) and Poincaré inequality, we deduce

$$
\begin{aligned}
V'(t) &= E'(t) + \epsilon\psi'(E(t))E'(t)\int_\Omega uu'dx\\
&\quad +\epsilon\psi(E(t))\int_\Omega [|u'|^2 - |\nabla u|^2 - ug(u')]dx\\
&= -\int_\Omega u'g(u')dx - \epsilon\psi'(E(t))\int_\Omega uu'dx\int_\Omega u'g(u')dx\\
&\quad +\epsilon\psi(E(t))\int_\Omega [-|u'|^2 - |\nabla u|^2]dx
\end{aligned}
$$

$$
\begin{aligned}
& +2\epsilon\psi(E(t))\int_\Omega |u'|^2dx - \epsilon\psi(E(t))\int_\Omega ug(u')dx \\
\leq\ & -\int_\Omega u'g(u')dx + \frac{1}{2}k\epsilon\psi'(E(t))\int_\Omega \Big(\frac{|u|^2}{k^2} + |u'|^2\Big)dx\int_\Omega u'g(u')dx \\
& -2\epsilon\psi(E(t))E(t) + 2\epsilon\psi(E(t))\int_\Omega |u'|^2dx - \epsilon\psi(E(t))\int_\Omega ug(u')dx.
\end{aligned}
$$

Moreover, by (2.6), we have $|u'|^2 \leq c_1^{-1}u'g(u')$ for $|u'| \geq 1$, and therefore taking into account that $\psi'(s)s$ is non-decreasing we deduce that

$$
\begin{aligned}
V'(t) \leq\ & -2\epsilon\psi(E(t))E(t) + [\epsilon k\psi'(E(0))E(0) - 1]\int_\Omega u'g(u')dx \\
& +2\epsilon c_1^{-1}\psi(E(t))\int_{[|u'|\geq 1]} u'g(u')dx + 2\epsilon\psi(E(t))\int_{[|u'|\leq 1]} |u'|^2dx \\
& -\epsilon\psi(E(t))\int_\Omega ug(u')dx \\
\leq\ & -2\epsilon\psi(E(t))E(t) \\
& +[\epsilon k\psi'(E(0))E(0) + 2\epsilon c_1^{-1}\psi(E(0)) - 1]\int_{[|u'|\geq 1]} u'g(u')dx \\
& +[\epsilon k\psi'(E(0))E(0) - 1]\int_{[|u'|\leq 1]} u'g(u')dx \\
& +2\epsilon\psi(E(t))\int_{[|u'|\leq 1]} |u'|^2dx \ \ (= I_1) \\
& -\epsilon\psi(E(t))\int_\Omega ug(u')dx \ \ (= I_2). \qquad (2.23)
\end{aligned}
$$

We now estimate I_1 and I_2 as follows. Let φ^* denote the dual of φ in the sense of Young (see [2, p. 64] for the definition). Then, by Young's inequality [2, p. 64] and Jensen's inequality [16], we deduce

$$
\begin{aligned}
I_1 =\ & 2\epsilon\mathrm{mes}(\Omega)\psi(E)\frac{1}{\mathrm{mes}(\Omega)}\int_{[|u'|\leq 1]} |u'|^2dx \\
\leq\ & 2\epsilon\mathrm{mes}(\Omega)\psi(E)\frac{1}{\mathrm{mes}(\Omega)}\int_{[|u'|\leq 1]} |u'|^{(p+1)/p}dx \ \ (\text{note that } |u'| \leq 1 \text{ and } p \geq 1) \\
\leq\ & 2\epsilon\mathrm{mes}(\Omega)\Big[\varphi^*(\psi(E)) + \varphi\Big(\frac{1}{\mathrm{mes}(\Omega)}\int_{[|u'|\leq 1]} |u'|^{(p+1)/p}dx\Big) \\
\leq\ & 2\epsilon\mathrm{mes}(\Omega)\varphi^*(\psi(E)) + 2\epsilon\int_{[|u'|\leq 1]} \varphi(|u'|^{(p+1)/p})dx\Big) \\
\leq\ & 2\epsilon\mathrm{mes}(\Omega)\varphi^*(\psi(E)) + 2\epsilon\int_{[|u'|\leq 1]} |u'|h(|u'|)dx \ \ (\text{use } (2.7)) \\
\leq\ & 2\epsilon\mathrm{mes}(\Omega)\varphi^*(\psi(E)) + 2\epsilon c_3^{-1}\int_{[|u'|\leq 1]} u'g(u')dx, \qquad (2.24)
\end{aligned}
$$

and

$$
\begin{aligned}
& \psi(E)\int_{[|u'|\le 1]}|g(u')|^{(p+1)/p}dx \\
\le\ & c_4^{(p+1)/p}\mathrm{mes}(\Omega)\varphi^*(\psi(E)) + c_4^{(p+1)/p}\int_{[|u'|\le 1]}\varphi(|c_4^{-1}g(u')|^{(p+1)/p})dx\big) \\
\le\ & c_4^{(p+1)/p}\mathrm{mes}(\Omega)\varphi^*(\psi(E)) + c_4^{(p+1)/p}\int_{[|u'|\le 1]}|c_4^{-1}g(u')|h(c_4^{-1}|g(u')|)dx \ \ \text{(use (2.7))} \\
\le\ & c_4^{(p+1)/p}\mathrm{mes}(\Omega)\varphi^*(\psi(E)) + c_4^{1/p}\int_{[|u'|\le 1]}u'g(u')dx. \qquad (2.25)
\end{aligned}
$$

Since we have assumed that

$$
u\in C^1([0,\infty),H_0^1(\Omega)), \qquad (2.26)
$$

it follows from (2.6), (2.7) and Sobolev's embedding theorem (see [1, p. 97]) that

$$
\int_\Omega u'g(u')dx \le c_2\int_{[|u'|\ge 1]}|u'|^{p+1}dx + \int_{[|u'|\le 1]}|u'||g(u')|dx<\infty, \quad \forall t>0. \qquad (2.27)
$$

Moreover, by (2.6), we have

$$
|g(u')|^{(p+1)} \ \le\ c_2[u'g(u')]^p, \quad |u'|\ge 1. \qquad (2.28)
$$

Thus, by Young's inequality [2, p. 64], for any positive constant λ, we have

$$
\begin{aligned}
|I_2| \ =\ & \epsilon\psi(E(t))|\int_\Omega ug(u')dx| \\
\le\ & \epsilon\psi(E(t))\|u\|_{H_0^1(\Omega)}\|g(u')\|_{H^{-1}(\Omega)} \\
\le\ & \alpha\epsilon\psi(E(t))\|u\|_{H_0^1(\Omega)}\Big(\int_\Omega|g(u')|^{(p+1)/p}dx\Big)^{p/(p+1)} \\
\le\ & \alpha\epsilon\psi(E(t))\|u\|_{H_0^1(\Omega)}\Big(\int_{[|u'|\le 1]}|g(u')|^{(p+1)/p}dx\Big)^{p/(p+1)} \\
& +\alpha\epsilon c_2^{1/(p+1)}\psi(E(t))\|u\|_{H_0^1(\Omega)}\Big(\int_{[|u'|\ge 1]}u'g(u')dx\Big)^{p/(p+1)} \\
\le\ & \alpha\epsilon\psi(E(t))\Big[\frac{p}{p+1}\lambda^{-(p+1)/p}\int_{[|u'|\le 1]}|g(u')|^{(p+1)/p}dx + \frac{1}{p+1}\lambda^{p+1}\|u\|^{p+1}_{H_0^1(\Omega)}\Big] \\
& +\alpha\epsilon c_2^{1/(p+1)}\psi(E(t))\Big[\frac{p}{p+1}\lambda^{-(p+1)/p}\int_{[|u'|\ge 1]}u'g(u')dx + \frac{1}{p+1}\lambda^{p+1}\|u\|^{p+1}_{H_0^1(\Omega)}\Big] \\
\le\ & \frac{\alpha\epsilon p}{p+1}c_4^{(p+1)/p}\mathrm{mes}(\Omega)\lambda^{-(p+1)/p}\varphi^*(\psi(E)) \ \ \text{(use (2.25))} \\
& +\frac{\alpha\epsilon}{p+1}2^{(p+1)/2}(1+c_2^{1/(p+1)})\lambda^{p+1}\psi(E(t))E^{(p+1)/2}(t) \\
& +\frac{\alpha\epsilon p}{p+1}c_4^{1/p}\lambda^{-(p+1)/p}\int_{[|u'|\le 1]}u'g(u')dx \\
& +\frac{\alpha\epsilon p}{p+1}c_2^{1/(p+1)}\lambda^{-(p+1)/p}\psi(E(0))\int_{[|u'|\ge 1]}u'g(u')dx. \qquad (2.29)
\end{aligned}
$$

It therefore follows from (2.23), (2.24) and (2.29) that

$$\begin{aligned} V'(t) \quad \leq \quad & -2\epsilon\psi(E(t))E(t) + 2\epsilon\mathrm{mes}(\Omega)\varphi^*(\psi(E)) \\ & +\frac{\alpha\epsilon p}{p+1}c_4^{(p+1)/p}\mathrm{mes}(\Omega)\lambda^{-(p+1)/p}\varphi^*(\psi(E)) \\ & +\frac{\alpha\epsilon}{p+1}2^{(p+1)/2}(1+c_2^{1/(p+1)})\lambda^{p+1}\psi(E(t))E^{(p+1)/2}(t) \\ & +(\epsilon k_1 - 1)\int_{[|u'|\geq 1]} u'g(u')dx + (\epsilon k_2 - 1)\int_{[|u'|\leq 1]} u'g(u')dx. \end{aligned} \tag{2.30}$$

where

$$k_1 = k\psi'(E(0))E(0) + 2c_1^{-1}\psi(E(0)) + \frac{\alpha p}{p+1}c_2^{1/(p+1)}\lambda^{-(p+1)/p}\psi(E(0)), \tag{2.31}$$

$$k_2 = k\psi'(E(0))E(0) + \frac{\alpha p}{p+1}c_4^{1/p}\lambda^{-(p+1)/p} + 2c_3^{-1}. \tag{2.32}$$

By the definition of the dual function in the sense of Young $\varphi^*(s)$ of the convex function $\varphi(s)$ of hypothesis (v), $\varphi^*(t)$ is the Legendre transform of $\varphi(s)$, which is given by (see [2, p. 61-62])

$$\varphi^*(t) = t\varphi'^{-1}(t) - \varphi[\varphi'^{-1}(t)]. \tag{2.33}$$

Thus, we have

$$\varphi^*(\psi(E)) = \psi(E(t))\varphi'^{-1}(\psi(E(t))) - \varphi[\varphi'^{-1}(\psi(E(t))). \tag{2.34}$$

This motivates us to make the choice

$$\psi(s) = \varphi'(as) \tag{2.35}$$

so that

$$\varphi^*(\psi(E)) = \varphi'(aE)aE - \varphi(aE)$$

where the constant a will be determined later. By condition (v), $\psi(s)$ satisfies the requirement we set at the beginning of the proof, that is, ψ and $\psi'(s)s$ are positive and increasing on $(0, +\infty)$. Taking

$$a = (2\mathrm{mes}(\Omega) + \frac{\alpha p}{p+1}c_4^{(p+1)/p}\mathrm{mes}(\Omega)\lambda^{-(p+1)/p})^{-1}, \tag{2.36}$$

and noting the definition (2.12), (2.13) and (2.17) of m_1, m_2 and ϵ, we deduce from (2.30) that

$$V'(t) \leq -\epsilon\varphi'(aE(t))E(t) - \epsilon m_1\varphi(aE(t)) + m_2\epsilon\lambda^{p+1}\varphi'(aE(t))E^{(p+1)/2}(t). \tag{2.37}$$

On the other hand, since $\varphi(s)$ and $\varphi'(s)$ are positive and increasing on $(0,\infty)$, it follows from Poincaré's inequality that

$$[1-\epsilon k\varphi'(aE(0))]E(t) \le V(t) \le [1+\epsilon k\varphi'(aE(0))]E(t). \tag{2.38}$$

Therefore, we deduce from (2.37) and (2.38) that

$$\begin{aligned} V'(t) \;\le\; & -\frac{\epsilon V(t)}{1+\epsilon k\varphi'(aE(0))}\varphi'\Big(\frac{aV(t)}{1+\epsilon k\varphi'(aE(0))}\Big) \\ & -\epsilon m_1\varphi\Big(\frac{aV(t)}{1+\epsilon k\varphi'(aE(0))}\Big) \\ & +m_2\epsilon\lambda^{p+1}\varphi'\Big(\frac{aV(t)}{1-\epsilon k\varphi'(aE(0))}\Big)\Big[\frac{V(t)}{(1-\epsilon k\varphi'(aE(0)))}\Big]^{(p+1)/2}. \end{aligned} \tag{2.39}$$

This is (2.9).

It remains to prove (2.10). We argue by contradiction. Suppose that $E(t)$ doesn't tend to zero as $t\to\infty$. Since $E(t)$ is decreasing on $[0,\infty)$, we have

$$E(0) \ge E(t) \ge \sigma > 0, \quad \forall t \ge 0, \tag{2.40}$$

and by (2.38), we have

$$bE(0) \ge V(t) \ge \beta > 0, \quad \forall t \ge 0. \tag{2.41}$$

Thus we have

$$\varphi'(aE(0)) \ge \varphi'\Big(\frac{aV(t)}{b}\Big) \ge \gamma > 0, \quad \forall t \ge 0. \tag{2.42}$$

Let $\lambda > 0$ be so small that

$$m_2\lambda^{p+1}\varphi'\Big(\frac{aV(t)}{c}\Big)\Big(\frac{V(t)}{c}\Big)^{(p+1)/2} \le m_1\varphi(a\beta/b), \;\; \forall t \ge 0. \tag{2.43}$$

It therefore follows from (2.9) that

$$V'(t) \le -\frac{\epsilon\gamma}{b}V(t), \;\; \forall t \ge 0, \tag{2.44}$$

which is in contradiction with (2.41). This completes the proof. □

Remark 2.2 The function φ which satisfies the conditions of Theorem 2.1 always exists. For example, we set

$$\bar\varphi(s) = \mathrm{conv}[s^{p/(p+1)}h(s^{p/(p+1)})], \tag{2.45}$$

where conv denotes the convex envelope of a function. Then we can take an increasing, convex and twice differentiable function $\varphi(s)$ such that $\varphi(s) \leq \bar{\varphi}(s)$. $\square$

Corollary 2.3 *Assume that $g \in C(\mathbf{R})$ satisfies all the conditions of Theorem 2.1. Suppose $\varphi(s) = s^{p/(p+1)}h(s^{p/(p+1)})$ is convex and twice continuously differentiable. Then the energy $E(t)$ of (1.1) satisfies the following decay rate:*

$$E(t) \leq 2V(t), \quad \textit{for } t \geq 0, \tag{2.46}$$

where $V(t)$ satisfies the following differential equation:

$$\begin{aligned} V'(t) &= -\frac{\epsilon(2p+1)a^{\frac{-1}{p+1}}}{(p+1)b^{\frac{p}{p+1}}} V^{\frac{p}{p+1}} h\Big((\frac{aV}{b})^{\frac{p}{p+1}}\Big) - \frac{\epsilon p}{(p+1)b}(\frac{a}{b})^{\frac{p-1}{p+1}} V^{\frac{2p}{p+1}} h'\Big((\frac{aV}{b})^{\frac{p}{p+1}}\Big) \\ &\quad + \frac{pm_2\epsilon\lambda^{(p+1)}}{p+1}\Big[(\frac{aV}{c})^{\frac{-1}{p+1}} h((\frac{aV}{c})^{\frac{p}{p+1}}) + (\frac{aV}{c})^{\frac{p-1}{p+1}} h'((\frac{aV}{c})^{\frac{p}{p+1}})\Big]\Big(\frac{V}{c}\Big)^{\frac{p+1}{2}}. \end{aligned} \tag{2.47}$$

Proof. Since

$$\varphi(s) = s^{p/(p+1)}h(s^{p/(p+1)}), \tag{2.48}$$

$$\varphi'(s) = \frac{p}{p+1}[s^{-1/(p+1)}h(s^{p/(p+1)}) + s^{(p-1)/(p+1)}h'(s^{p/(p+1)})], \tag{2.49}$$

substituting (2.48) and (2.49) into (2.9), we obtain (2.47). $\square$

3 Examples

In this section, we give three examples to illustrate how to derive from our general result the usual exponential or polynomial decay rate and the logarithmic decay rate for the exponenatially degenerate damping. In what follows, by ω we denote various positive constants that may vary from line to line.

Example 1. *Exponential Decay Rate.* Let $g(s) = \ell s$ and $p = 1$, where ℓ is a positive constant. Then $h(s) = \ell s$ as well. In this case, all the assumptions of Corollary 2.3 are satisfied and (2.47) becomes

$$V'(t) = -\omega V(t), \tag{3.1}$$

where ω is a positive constant. Thus, as usual, we obtain an exponential decay rate. $\square$

Example 2. *Polynomial Decay Rate.* Assume $g(s) = \ell|s|^{q-1}s$ with $q > 1$ and $\ell > 0$. Then $h(s) = \ell s^q$ and $p = 1$, $q > 1$. Then (2.47) becomes

$$V'(t) = -\omega[V(t)]^{(q+1)/2}, \tag{3.2}$$

which, as usual, implies the polynomial decay rate

$$E(t) \leq C(E(0))t^{-2/(q-1)}, \quad \forall t > 0. \tag{3.3}$$

$\square$

Example 3. *Logarithmic Decay Rate.* Let $p = 1$ and $g(s) = s^3 e^{-\frac{1}{s^2}}$ near the origin. Let

$$h(s) = s^3 e^{-\frac{1}{s^2}}, \quad s > 0. \tag{3.4}$$

Then, by (2.47), V satisfies

$$V'(t) \leq -\omega V^2 e^{-\frac{b}{aV}}, \tag{3.5}$$

which is the same as

$$\left(e^{\frac{b}{aV}}\right)' \geq \frac{b\omega}{a}. \tag{3.6}$$

Solving the inequality, we obtain the logarithmic decay rate

$$V(t) \leq \frac{b}{a}\Big[\log\Big(\frac{b\omega}{a}t + e^{\frac{b}{aV(0)}}\Big)\Big]^{-1}. \tag{3.7}$$

We deduce that

$$E(t) \leq c_1/log(c_2 t) \tag{3.8}$$

for suitable positive constants c_1 and c_2.

In these exmaples the decay rate is totally determined by h. Thus g does not need exactly the function we have given. Any other function saisfying the conditions of Theorem 2.1 for this h would lead to the same decay rate.

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