A Thermoacoustic Approach to Exploring Resonance

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Acknowledgements

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1 Introduction to Resonance

When a person blows air over the top of a bottle, the air blown into the bottle causes the air inside the bottle to oscillate. If the person blows too hard or too softly, the air inside the bottle oscillates, but the amplitude of the oscillation is small. However, when a person blows into a bottle "just right," the amplitude becomes increasingly large, and the bottle will produce sound. This production of sound is an example of resonance. Mechanical resonance occurs when a system is driven at a frequency that is equal to the system's natural frequency, which is dependent upon a system's physical characteristics. When a system is driven at its resonance frequency, energy is most efficiently transfered to the system, and if the system allows for movement, the system will oscillate at ever-increasing amplitudes.[1] Other examples of mechanical resonance include the production of musical tones from wind instruments, the shattering of a crystal goblet by an opera singer, and an off-balance washing machine.

Mechanical resonance has very important implications for design and engineering. If a structure or piece of equipment is designed without considering the effects of resonance, then the equipment or structure's stability may be compromised. The "Galloping Gertie" bridge in Tacoma, Washington, is an infamous example of such neglect. Due to the design of the bridge, the wind was able to drive the bridge to resonate, causing the bridge to vibrate at extremely large amplitudes. In 1940 the bridge's vibrating amplitude was too great for the structure's support, and the bridge collapsed.

London's Millenium Bridge is a modern example of the effects of mechanical resonance. During the bridge's opening on June 10, 2000, an estimated 80,000 to 100,000 pedestrians crossed the bridge with no more than 2,000 pedestrians on the bridge at one time. As the pedestrians crossed the bridge, their footsteps contributed to the bridge's natural vibrations. These lateral vibrations were observed to dampen as the number of people crossing the bridge or the movement of the pedestrians decreased. In order to dampen these vibrations, engineers installed thirty-seven viscous dampers as well as twenty-six vertical dampers as a precaution.[2]

The Tacoma Narrows Bridge and London's Millenium Bridge[1] are two examples of structures being built without considering the effects of mechanical resonance upon them. For the Tacoma Narrows Bridge the effects of resonance had very severe consequences that ultimately led to the destruction of the bridge. London's Millenium Bridge is fortunate in that the effects of resonance were not as dire; however, more time and money had to be spent in determining a solution to the bridge's increasing vibrations to ensure the bridge's integrity. The resulting consequences for both bridges could have been eliminated had the effects of resonance been more effectively studied prior to their construction. However, it is not necessary to construct full-scale models of the bridges in order to study the effects of resonance. Studying scale models or analogous systems can be just as efficient and effective in determining the nature of resonance in various systems. Therefore, it is imperative to study smaller resonant systems in order to better understand the effects of resonance upon larger systems.

2 Traditional Approach to Resonance

In several physical science courses, mechanical resonance is traditionally introduced via a spring-mass system. In this system, a spring is vertically attached to a driving paddle that can be adjusted to drive the spring at various frequencies. A mass is attached to the bottom of the spring. The natural or resonance frequency for this spring-mass system is

[1] Prior to London Millenium Bridge's opening, tests were conducted to determine how the bridge would behave due to wind and vertical pedestrian excitation. However, lateral pedestrian excitation was not considered.[2]

$$\nu_o = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{1}$$

where ν_o is the resonance frequency, k is the spring constant and m is the mass. When the spring-mass is driven below its resonance frequency, the spring-mass and the paddle are oscillating in phase, meaning that the paddle and the spring-mass are moving together in the same direction as shown in Figure 1. When the spring is driven above its resonance frequency, the spring-mass and the paddle are oscillating out of phase, meaning that the paddle and the mass-spring are moving in opposite directions of one another as shown in Figure 2.

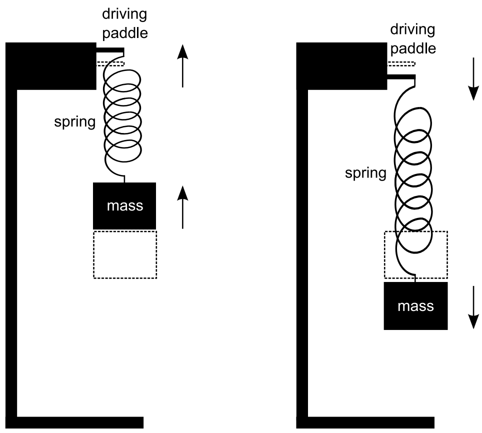


Figure 1: Driving paddle and mass moving in phase.

When the spring-mass is driven at its resonance frequency by the paddle, the paddle and the spring-mass move together in the same direction, and the amplitude of the spring-mass's vibrations get increasingly large.

The amplitude of the spring-mass's oscillation will be affected by the mechanical resistance in the system. The main source of resistance in the spring-mass system is the air's resistance to the moving mass. The quality factor for a spring-mass system describes the amplitude of the spring-mass's oscillation in terms of the system's resonance frequency and resistance and is given by:

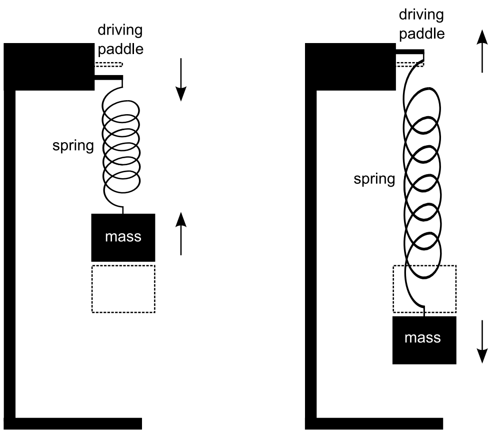


Figure 2: Driving paddle and mass moving out of phase.

$$Q = \frac{\omega_o m}{R} \tag{2}$$

In this equation Q represents the quality factor, ω_o represents the resonator's natural angular frequency[2] , and R is the resistance of the system.[3] A system with a high quality factor has little resistance and a large oscillating amplitude at resonance whereas a system with a low quality factor has more resistance and a dampened response at resonance. The quality factor is also associated with the shape of the resonance peak for a system. A system with a high quality factor will have a sharp peak whereas a system with a low quality factor will have a broad peak.

The motion of the mass at the end of the spring can be described by Newton's Second Law, which states that the sum of the forces acting upon an object are equal to its mass times its acceleration.

$$\Sigma\vec{F} = m\vec{a} \tag{3}$$

There are three forces acting on the mass in the driven, damped spring-mass system. The first force is the force driving the mass to oscillate and is denoted $f(t)e^{i\omega t}$. The

[2] $\omega_o = 2\pi\nu_o$

sinusoidal component, $e^{i\omega t}$, mathematically describes the paddle's sinusoidal motion, which is dependent upon the angular frequency, ω, and time, t. The second force resists the motion of the mass as it moves through the air and is represented by Rv, where R is the resistance and v is the velocity of the moving mass. The third force is the restoring force of the spring, kx, where k is the spring constant and x is the displacement[3]. The forces due to the air resistance and the restoring nature of the spring are negative because both forces oppose the motion of the mass.

However, there is a fourth force acting upon the mass- gravity. Before the mass is driven by the oscillating paddle, the mass is attached to the spring, and the force of gravity pulls the mass downward, stretching the length of the spring. Eventually, the mass's weight will be offset by the upward restoring force of the spring, and the mass ceases its downward motion. When this situation occurs, the system is said to be at equilibrium. After the system reaches equilibrium, the driving paddle is then used to oscillate the mass about its equilibrium position. Therefore, the force due to gravity is not considered when inserting the forces acting upon the mass into Equation 3.

Inserting the three forces that act upon the mass into Newton's Second Law yields:

$$f(t)e^{i\omega t} - Rv - kx = ma \tag{4}$$

Appendix A demonstrates how to mathematically manipulate this equation to give the ratio of the displacement, x, to the maximum dispacement, x_{max}, at resonance as a function of frequency, ν, and the quality factor, Q, for the system.

$$\frac{x}{x_{max}} = \frac{1}{\sqrt{1 + \left(\frac{\nu_0}{\nu}\right)^2 Q^2 \left[1 - \left(\frac{\nu}{\nu_0}\right)^2\right]^2}} \tag{5}$$

An educational demonstration of a driven, damped spring-mass system can be

[3]This statement is also referred to as Hooke's Law

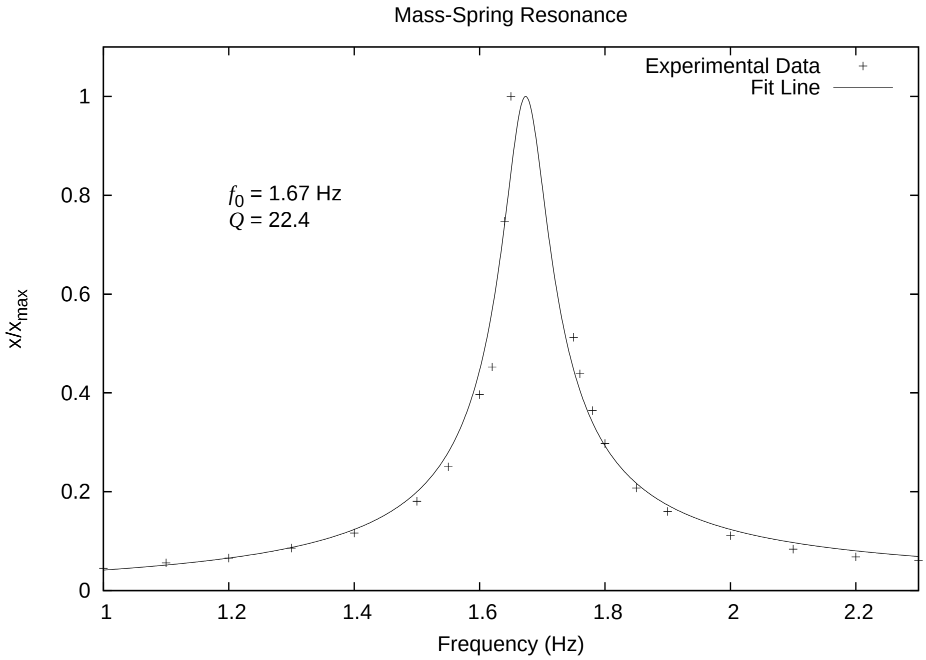


Figure 3: Resonance curve for a driven, damped spring-mass system. This system has a resonance frequency of 1.67 Hz (represented here as f_o) and a quality factor of 22.4.

constructed from a PASCO WA-9857 String Vibrator, a PASCO WA-9867 Sine Wave Generator, a spring, and a hanging mass. This demonstration is similar to the spring-mass systems shown in Figures 1 and 2. The frequency at which the string vibrator (driving paddle) oscillated and the amplitude of the oscillation could be controlled using the sine wave generator. The spring-mass was driven from 1 to 2.3 Hertz[4], and the displacement of the mass was measured at every tenth of a Hertz. The ratio of the displacement to the maximum displacement at resonance was plotted as a function of frequency, and the data was fitted to Equation 5. From the fit, the resonance frequency and the quality factor for this system were determined to be 1.67 Hertz and 22.4, respectively. One can see that as the system approached resonance, the spring-mass's amplitude of oscillation increased until it reached its maximum at 1.67 Hertz. As the system was driven at frequencies beyond the resonance frequency, the

[4] oscillations/second

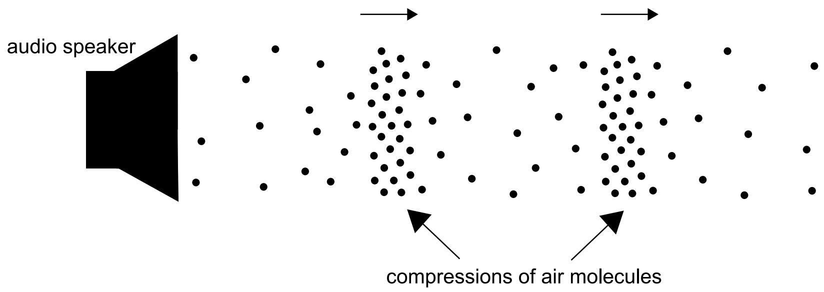


Figure 4: Compressions of air molecules moving out from the vibrating loudspeaker.

amplitude of the oscillation decreased. Although the resonance peak appears to be fairly narrow, the quality factor of 22.4 for the system indicates some resistance in the system, which is due to the resistive force of the air against the motion of the mass.

3 Thermoacoustic Approach to Resonance

3.1 Basics of Acoustics

Consider an audio speaker that is producing sound. A loudspeaker's vibrating membrane compresses the air molecules closest to the membrane, and this disturbance moves outward from the speaker as a traveling sound wave as seen in Figure 4. Therefore, sound is simply defined as a mechanical disruption of molecules due to pressure changes.[4]

Sound waves are physical traveling waves and are characterized by their frequency and amplitude. Frequency refers to the number of waves that pass a point in space within a second, and the amplitude is a measure of the wave's "strength." A person with impeccable hearing can detect a range of sound waves with frequencies of 20 Hertz (20 waves per second) to 20,000 Hertz; however, waves below 20 Hertz (infrasonic) and above 20,000 Hertz (ultrasonic) are still classified as sound waves, despite the fact that they are not heard by people.[4] Sound waves may also travel through various mediums, including gases, liquids, and solids. Therefore, when a person speaks of sound scientifically, she must take care to understand that sound encompasses waves that are not limited to human hearing.

The study of sound, or acoustics, branches over several disciplines, including engineering. One aspect of acoustical engineering focuses on various techniques to enhance or dampen sound waves that can propagate within buildings and structures. Because sound waves can be amplified when a system is driven at its natural frequency, acoustical engineers must find solutions to dampen resonating systems; however, the effects of acoustic resonance are often considered after construction is complete. Like the mechanical engineers who researched methods to dampen the vibrations on London's Millenium Bridge, acoustical engineers must find solutions to limit the effects of acoustic resonance. These solutions require more money and time to be invested in the building or structure than initially expected.

Because the principles of mechanical resonance and acoustical resonance are identical, the effects of mechanical resonance on a large system can be studied with the use of a smaller acoustic system. An acoustic system that may offer a unique means in which to explore resonance is a thermoacoustic engine. By exploring resonance via thermoacoustic engines, students can study the effects of mechanical resonance on a system that requires very few or no moving parts and can be built on a relatively small scale.

3.2 Thermoacoustic Engines

Thermoacoustics is the study of how oscillating gas particles and their interactions with solid boundaries might be utilized to design new energy-based technologies, such as engines and refrigerators.[5] Thermoacoustic engines are ideal in that they have very few or no moving parts and do not release harmful agents into the environment like traditional engines.[5] A thermoacoustic engine is based upon the same scientific principles as a thermodynamic engine. When heat is put into a thermoacoustic engine, the engine produces work in the form of sound with an amount of waste heat rejected into the cooler environment. However, unlike a traditional heat engine, a thermoacoustic engine contains no moving parts, thus making it a desirable alternative to tradition engines due to its high reliability and low cost.[5] While thermoacoustics has many practical applications regarding the design and fabrication of technical equipment, the field may also provide a new means in which to study the effects of resonance.

3.2.1 Heat Engines

Heat engines convert heat into usable energy. In science, one often calls this usable energy *work* (W). Figure 5 illustrates a basic diagram of the process of converting heat into work. The engine first absorbs heat, Q_h, from a hot reservoir. The engine then transforms that heat into work, and a certain amount of waste heat, Q_c, is ejected into the cold resevoir (i.e. the environment). This process describes an ideal engine, in which internal friction and heat losses are considered negligible.[6]

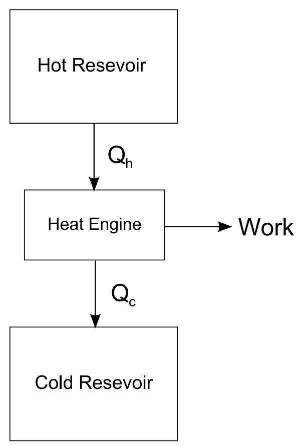


Figure 5: Energy transfer in a heat engine.

In all physical processes the law of conservation of energy must hold, meaning that the energy that is first introduced into the process must equal the exiting energy. For the heat engine, this conservation of energy can be mathematically described by the equation:

$$Q_h = W + Q_c \tag{6}$$

An engine's performance is described by its efficiency, e. An engine is efficient if it can maximize the amount of work transformed from an available amount of energy. [6]

$$e = \frac{\text{usable energy}}{\text{heat put into engine}} \tag{7}$$

However, in thermal physics texts, the efficiency for a heat engine is given by the equation:

$$e = \frac{\text{Work}}{Q_h} \tag{8}$$

The second law of thermodynamics prevents the efficiency of the engine from ever equalling one[5]. A certain amount of waste heat *must always* be ejected during the process. For example, consider a car's engine. A spark from the spark plug ignites a compressed fuel-air mixture, and the mixture explodes. This ignition from the spark serves as the introduction of heat into the engine. The gas expands due to the increase in temperature[6], and this expansion will exert a force on the piston, causing the piston to move another piece of machinery. However, not all of the heat will be used in the process, and the waste heat as well as other combustion products that resulted from the reaction will be released into the environment as exhaust.[6]

3.2.2 Theory of Thermoacoustic Engines

In order to better understand the actual thermoacoustic process, one has to look at how the gas particles within a thermoacoustic engine are behaving. Consider an

[5] Kelvin's statement of the Second Law of Thermodynamics: "No process is possible whose sole result is the complete conversion of heat into work."[7]

[6] This expansion is mathematically described by the ideal gas law, $PV = NkT$, where P is the pressure, V is the volume, N is the number of gas particles, k is Boltzmann's constant, and T is the temperature. Because temperature is directly proportional to volume, an increase in the temperature will result in an increase in volume.

empty pipe with one end closed and the other end open to the environment. The gas particles at the closed end will experience greater pressure than the molecules at the open end because the particles are constrained to a limited area. The gas particles at the open end will have more room to move and will therefore experience greater changes in their velocity. In the thermoacoustic process gas particles must experience both pressure and velocity changes.

Now consider a porous material (e.g. several soda straws taped together) that is inserted into the middle of the pipe as shown in Figure 6. A gas particle located in the porous material will experience both pressure and velocity changes because of its location in the pipe. From Euler's equation[7] it is possible to prove that the middle of the pipe is the proper location for the stack by observing the changes in pressure that occur in the pipe as it is driven by an external source:

$$u_{AC} = \left| \frac{1}{\omega_o \rho_o} \frac{\Delta P_o}{\Delta x} \right| \tag{9}$$

In this equation u_{AC} is the acoustic velocity, ρ_o is the ambient air density, and ΔP_o is the change in acoustic pressure over a certain length, Δx. ΔP_o increases moving towards the opening of the pipe, which means that the acoustic pressure amplitude is significantly decreasing as one approaches the opening. From Euler's equation, one can determine that a gas particle at this location will oscillate back-and-forth due to changes in its velocity. Towards the closed end of the pipe, ΔP_o decreases because the acoustic pressure is at its greatest towards the closed end of the pipe and will not significantly change over a given distance. Therefore, the velocity of the gas particles decreases, as well. However, a gas particle at this location will expand and contract due to experiencing pressure oscillations that occur between the acoustic and ambient air pressures. Therefore, a gas particle located in the middle

[7] A derivation of Euler's equation can be viewed in Appendix B

will oscillate back and forth due to the velocity changes caused by the difference in acoustic pressure amplitudes between two locations and will expand and contract due to the difference between the acoustic and ambient air pressures.

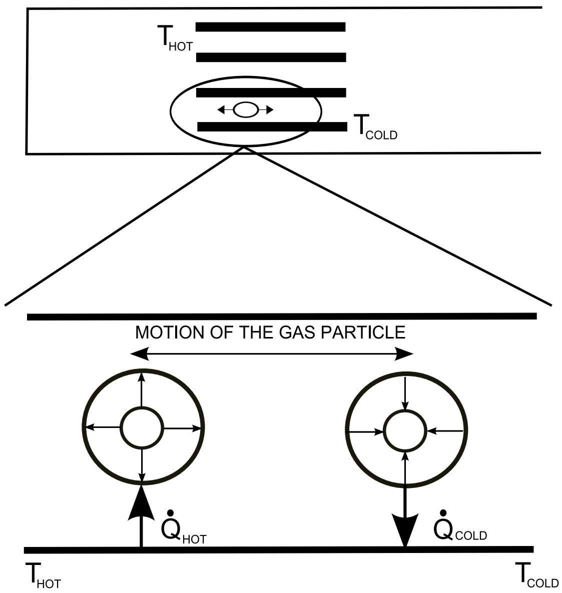


Figure 6: A closed-open pipe with a porous material inserted into the middle of the pipe.

If heat is added such that the temperature at one end of the stack, T_{HOT}, is greater than the other end of the stack at temperature T_{COLD}, then the gas particle will now transfer the added heat energy down the stack. If a particle is cooler than the warmer end of the stack, it will absorb heat, Q_{HOT}. The particle expands due to the pressure decrease as it travels a very small distance in the direction of the open end of the pipe. If the particle comes into contact with a cooler portion of the material, it will transfer its heat, Q_{COLD}, to the material. As the particle travels towards its orginal position, it will contract due to the increase in pressure. The particle will continue to repeat this heat transfer for as long as a temperature difference is generated across the porous material. Figure 7 shows a graphical representation of this heat transfer cycle for the particle.

All of the gas parcels located within the material act as a bucket brigade, perform-

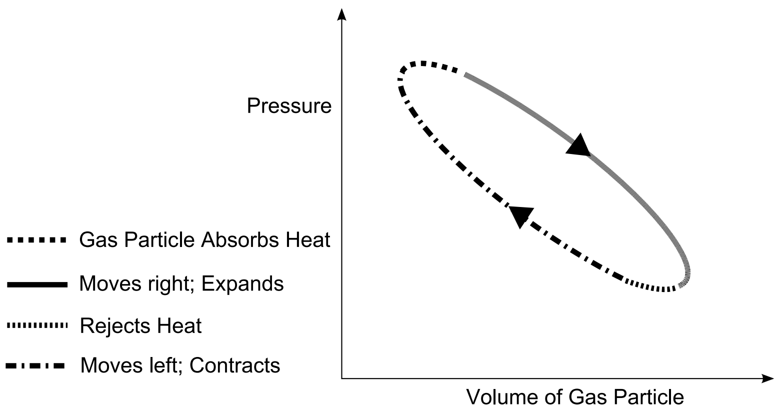


Figure 7: The heat transfer cycle for a gas particle located within the stack.

ing work on the system by transferring heat to the cooler end of the material.[5] If the temperature difference is large enough, the gas particles will perform enough work on the system for the system to overcome internal resistance and produce sound. This production of sound is referred to as onset.

4 Constructing Thermoacoustic Engines from Helmholtz Resonators

4.1 Helmholtz Resonators

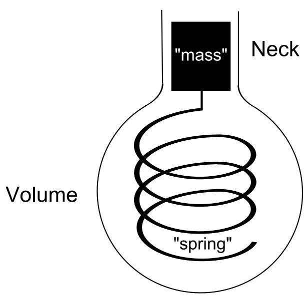


Figure 8: The behavior of air within a Helmholtz flask.

A bottle is a special kind of resonator known as a Helmholtz resonator. A Helmholtz resonator consists of a hollow neck attached to an empty volume.[3] In order to understand how the air inside a Helmholtz flask behaves, consider a mass bouncing on a spring. The air oscillating in the neck is comparable to the mass, and the air inside the cavity is comparable to the spring (see Figure 8).

When a person blows over the top of a soda bottle, she forces, or drives, the mass of air inside the neck to push down upon the air in the volume. The air inside the volume compresses like a spring, and the pressure increases. When the air inside the volume can no longer be compressed, the air will push back upon the air in the neck. This process will continue for as long as the person blows air over the opening of the bottle. Therefore, as long as the air within the flask is driven, the air within the bottle will oscillate.

When a person drives the air within a soda bottle at the right speed, the bottle will produce sound or resonate. The tone that is produced is related to the resonator's natural frequency, which is dependent upon the geometry of the flask. The equation:

$$\nu_o = \frac{c}{2\pi}\sqrt{\frac{S}{L'V}} \tag{10}$$

allows one to calculate the natural frequency. In this equation, ν_o is the natural frequency, c is the speed of sound in the air, S is the the cross-sectional area of the neck, L' is the effective length of the neck,[8] and V is the volume. [3] If a person can calculate the speed of sound and measure the dimensions of the resonator, she can easily determine the frequency at which the resonator will produce sound.

If the mass of air within the neck experiences friction, then its motion will become damped due to this added resistance. In fact, the air oscillating in the neck does experience friction due to the air molecules rubbing against the walls of the neck. This resistance along with the flask's natural frequency defines a quality factor for the flask. For a system with large resistance, the quality factor is quite small, but for a system with small resistance, the quality factor is large. The air inside a Helmholtz flask with large resistance will experience a very weak response when driven at its natural frequency whereas a flask with little or no resistance will experience a very

[8]The effective length is related to the length of the neck by the equation: $L' = L + 1.5a$, where a is the radius of the neck's opening. The effective length accounts for the shape of the opening of the resonator. In this demonstration the opening is unflanged, meaning that the opening of the flask does not have a broad ridge.

strong response. Therefore, a low quality factor is associated with a weak resonance, and a high quality factor corresponds to a strong resonance. Because of its well-defined geometry and relatively small size, a Helmholtz flask is ideal for studying the effects of resonance.

Because the air moving within the Helmholtz flask is comparable to a driven, damped spring-mass system, one can describe the displacement of the air within the Helmholtz flask's neck as it is driven at a certain frequency using Equation 5. However, the displacement of the mass of air is not easily measureable, and it makes more sense to measure the pressure of the air when driven at a certain frequency. Appendix C describes how to transform Equation 5 from describing a ratio of displacements to a ratio of pressure amplitudes. The resulting equation is:

$$\frac{A}{A_{max}} = \frac{1}{\sqrt{1 + \left(\frac{\nu_0}{\nu}\right)^2 Q^2 \left[1 - \left(\frac{\nu}{\nu_0}\right)^2\right]^2}} \tag{11}$$

4.2 Theory of Thermoacoustic Engines Constructed from Helmholtz Resonators

Adding a porous ceramic material into the neck of a Helmholtz flask will increase the system's resistance even more due to the friction between the oscillating air particles and the walls of the pores. This added resistance will decrease the quality factor for the system; however, the reduced quality factor of

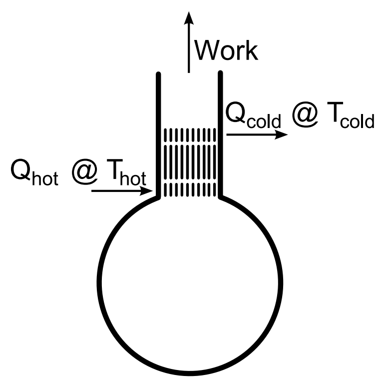


Figure 9: Heat enters the thermoacoustic engine and is then transformed into work and waste heat.[8]

this system can be increased by ap-

plying a temperature difference across the stack. As the temperature difference across the stack increases, then the quality factor of the system increases, meaning that the resistance is decreasing. This increased quality factor corresponds to a strong response from the system. If the system's resistance is eliminated, then the system produces work in the form of sound.[5]

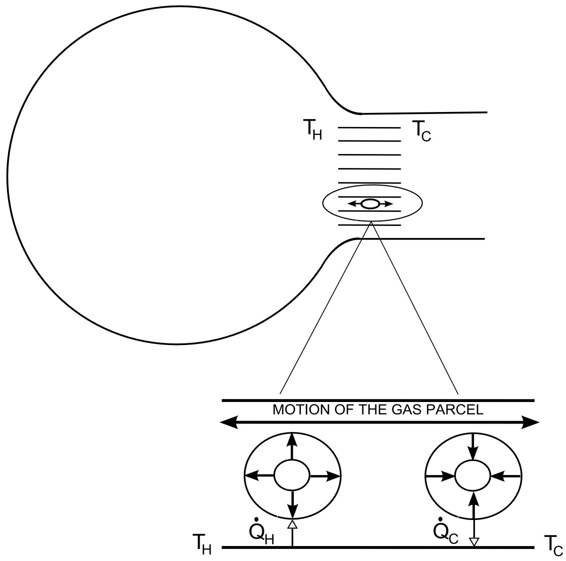


Figure 10: Motion of a gas parcel within a Helmholtz thermoacoustic engine

The motion of the gas particles in the stack behave identically to the gas particles moving in a stack placed inside an open-closed pipe as discussed in the Section 3.2.2. The gas particles are oscillating back and forth due to the pressure and velocity changes, and when heat is introduced at one end of the stack, the particles will transfer heat from the warmer to the cooler end of the stack. When the particles produce enough work to overcome the resistance of the system, then the engine will produce sound.

The main difference between a thermoacoustic engine constructed from an open-closed pipe and a thermoacoustic engine constructed from a Helmholtz resonator is the placement of the stack. For an open-closed pipe, the stack must be placed in the middle of the pipe for the gas particles within the stack to experience both pressure and velocity changes. However, due to the unique geometry of the Helmholtz resonator, the ideal location for the stack is at the junction of the resonator's neck and volume. The process for which this location was determined is described further in Section 5.

5 Experimental Setup and Methodology

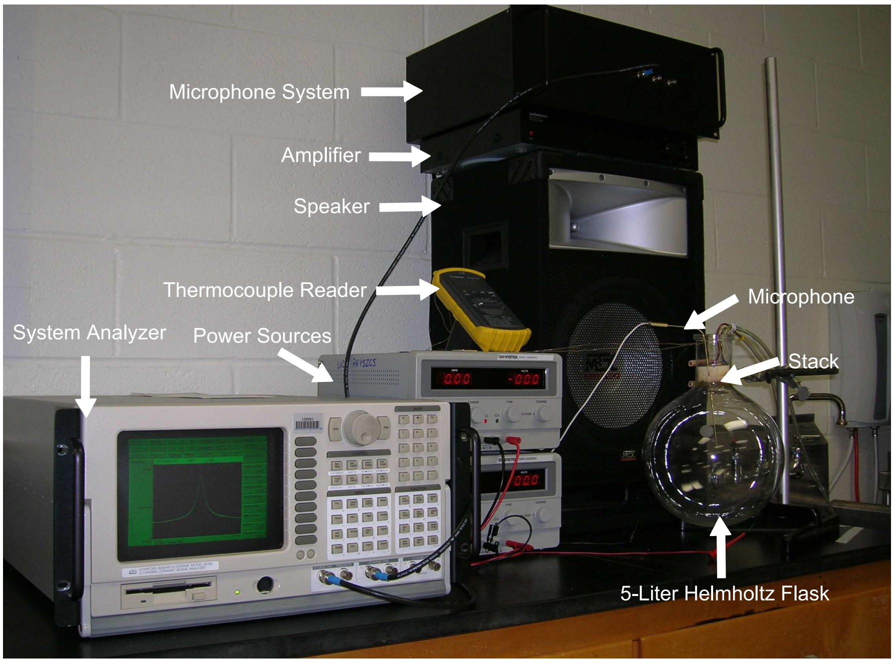


Figure 11: Experimental setup for observing resonance in a thermoacoustic engine constructed from a 5000 cm^3 Helmholtz resonator.

The following experiment investigated the acoustic response of a thermoacoustic engine constructed from a 5000 cm^3 Helmholtz flask as it approached resonance. Before the engine could be constructed, it was necessary to determine the resonance frequency and the quality factor for the empty flask. The neck of the Helmholtz flask used in this experiment has a cross-sectional area of 2.03×10^{-3} m^2 and length

8.90×10^{-2} m for its neck. Over the course of the experiment, the air inside the Helmholtz flask's neck was driven over multiple ranges of frequencies. The "driving force" for this experiment was provided by a MTX Audio TP112 speaker. The flask was placed approximately twenty centimeters in front of the speaker with the opening of the engine aligned with the middle of the speaker. A Stanford Research Systems SR785 signal analyzer controlled the frequency at which the speaker drove the flask. The signal from the signal analyzer was first provided to an AudioSource Model AMP One/A, amplifier, which increased the signal's amplitude or strength to the speaker. An Endevco Model 8510B-1 microphone was attached to a copper tube with an inner diameter of 0.014 in (0.03556 cm) and an outer diameter of 0.0625 in (0.15875 cm). The microphone tube was placed inside the flask such that the microphone would measure the pressure changes occurring in the middle of the flask's cavity as the flask was driven over a range of frequencies. The microphone's frequency and peak-to-peak voltage measurements were recorded by the signal analyzer, which also displayed the amplitude of the microphone's peak-to-peak voltage measurement as a function of frequency. One could determine the resonance frequency and the quality factor of the empy flask by fitting the measurements to Equation 11 for the ratio of pressure amplitudes of the air driven inside the Helmholtz flask.

After the resonance frequency and the quality factor had been determined, it was necessary to determine the proper position of the stack inside the flask so that the flask could be converted to a thermoacoustic engine. The air inside of the stack needed to experience both pressure and velocity changes in order for the engine to work, so pressure measurements were made by moving the end of the microphone tube to various vertical positions in the flask as the empty flask was driven at its resonance frequency. The setup for this portion of the experiment can be viewed in Figure 12. It was experimentally determined that the lower portion of the flask's neck

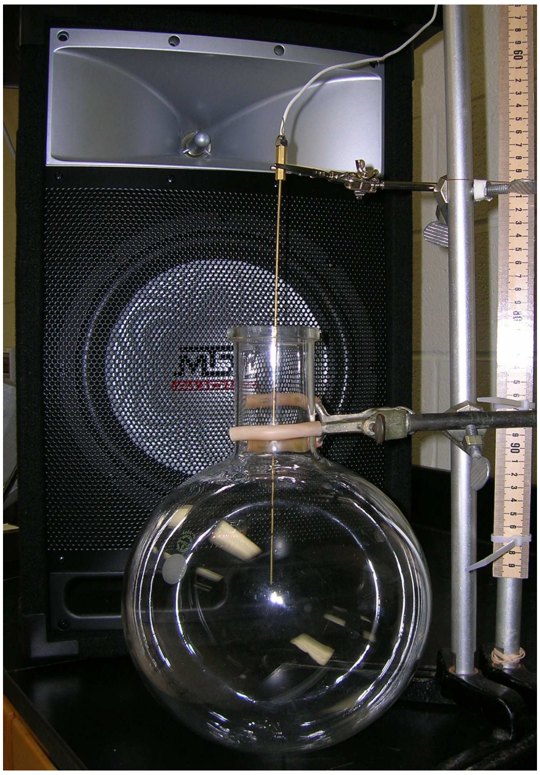


Figure 12: Experimental setup for measuring the pressure at different vertical positions inside the $5000\ cm^3$ Helmholtz resonator.

was the ideal location for the stack.

The porous ceramic substrate or stack used in this experiment had 324 pores per square inch (approximately 50.2 pores per square centimeter), a diameter of 4.70 centimeters, and a depth of 2.39 centimeters. Small grooves were cut onto one face of the stack in preparation of wiring that face with 2.571 Ω/foot (78.4 Ω/centimeter) nichrome wire. The nichrome wire served as the hot heat exchanger in this experiment. Both ends of the nichrome wire were connected to fifty-centimeter-long strands of 26-gauge wire. Later in the experiment, both wires would be connected to two parallel GW INSTEK Model GPR-3060D labaratory DC power supplies that would provide the nichrome wire with electrical power. An Omega Type K thermocouple was fed through the middle pore and adhered inside the pore just at the opening on the face of the stack with the nichrome wire. This thermocouple would eventually

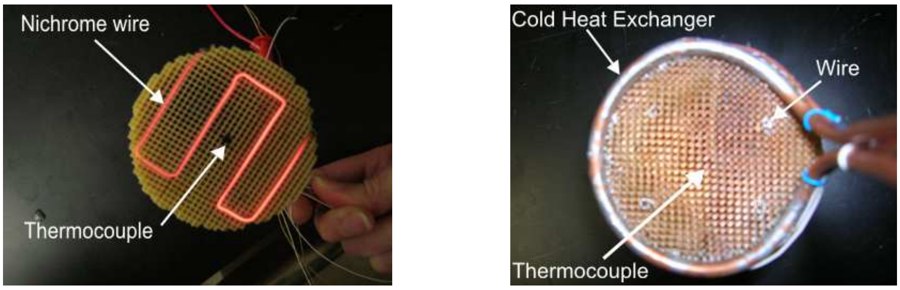


Figure 13: Bottom and top views of the porous ceramic substrate. The bottom view shows the hot heat exchanger whereas the top view shows the cold heat exchanger.

measure the temperature on this face of the stack. The hot heat exchanger can be seen in Figure 13.

On the other face of the stack, a copper cold heat exchanger was adhered using four strands of 30-gauge wire. The cold heat exchanger consisted of a circular portion of copper mesh approximately the diameter of the stack. A 0.125 in (0.3175 cm) outer diameter copper tube lined the outer rim of the mesh. During the course of the experiment, ice water would be pumped through the tube in order to achieve large temperature differences across the stack. Beneath the cold heat exchanger another thermocouple was placed just within the opening of the middle pore in order to measure the temperature on this face of the stack. The cold heat exchanger can be viewed in Figure 13.

The stack was inserted into the bottom portion of the flask's neck with the hot heat exchanger facing the cavity. After the stack had been placed, ice water was pumped through the cold heat exchanger, the temperatures on both sides of the stack were allowed to equilibriate, and the flask was swept over a range of frequencies. At this time no power was supplied to the nichrome wire. Measurements for frequency and peak-to-peak voltage were recorded by the system analyzer, and the resonance frequency and quality factor for the 5000 cm^3 flask with the stack in its neck were calculated using Equation 11.

After observing how the insertion of the stack affected the system's resonance curve, the next step was to investigate how increasing the temperature difference across the stack would affect the system's response. A current of one ampere was initially fed through the nichrome wire. The system was given ten minutes for the temperatures on both sides of the stack to equilibriate. Once the system's temperatures had equilibrated, it was swept over a range of frequencies, while the system analyzer recorded the frequency and peak-to-peak voltage measured by the microphone. The data was fitted, and the resonance frequency and quality factor were determined for the system. The power supplied to the nichrome wire was increased by one ampere increments, and this procedure was repeated for each increase in power until the system reached onset. Overall, seven different measurements were made for seven different temperature differences across the stack.

6 Experimental Results

6.1 Resonance Curve for Empty Helmholtz Flask

The empty five liter Helmholtz flask was swept over a range of frequencies. Figure 14 displays the ratio of the pressure amplitude to the maximum pressure amplitude at resonance on the y-axis and the range of frequencies on the x-axis. The data points are fitted to equation Equation 11 for the ratio of amplitudes for the air driven inside of the Helmholtz flask. According to Equation 10 for the resonance frequency of a Helmholtz flask, the resonance frequency is calculated to be approximately 97 Hz. As shown from the graph, the microphone measures maximum acoustic pressure amplitude when the flask is subjected to a driving frequency of 97.5 ± 0.003 Hz. The quality factor for this resonance curve was determined numerically to be 70.1 ± 0.184 . This large quality factor corresponds to a fairly sharp peak in the resonance curve

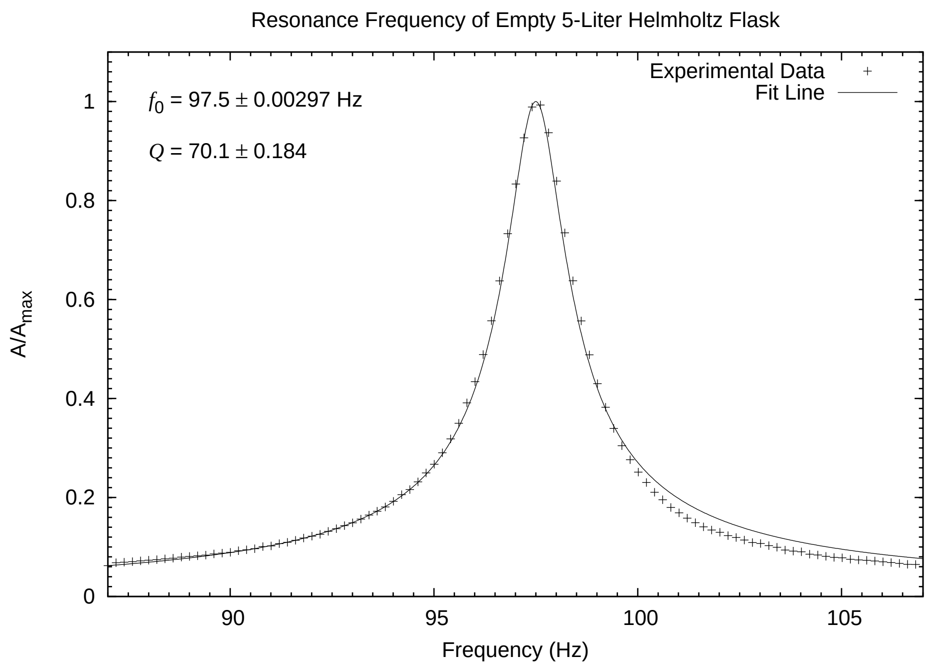


Figure 14: Resonance curve for an empty 5000 cm^3 Helmholtz flask.

and indicates low mechanical resistance. The resistance that is in the system is due to the oscillating mass of air in the Helmholtz flask's neck rubbing against the sides of the neck.

6.2 Pressure inside Helmholtz Flask

Figure 15 displays the vertical positions of the flask. The opening of the flask corresponds to zero centimeters, the neck of the flask extends from zero to nine centimeters, and the cavity extends from nine to thirty-one centimeters. Measurements for the pressure at various positions within the flask can be

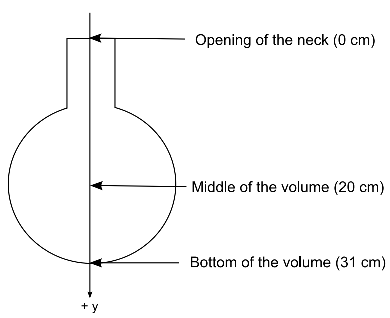


Figure 15: Vertical positions used for pressure measurements within the Helmholtz flask.

seen in Figure 16. From the opening of

the flask to the bottom of the neck, the pressure varies with position. However, from the bottom of the neck to the bottom of the flask, the pressure is only slightly changing. Referring to Equation 9, one can see that the air within the neck will experience mainly velocity oscillations, which is comparable to a mass oscillating about its equilibrium position, whereas the the air within the cavity will experience mainly pressure oscillations, which is comparable to a spring compressing and extending. Because the air within the stack needs to experience both velocity and pressure oscillations, one can determine from the graph that the ideal location for the stack is at the bottom of the flask's neck.

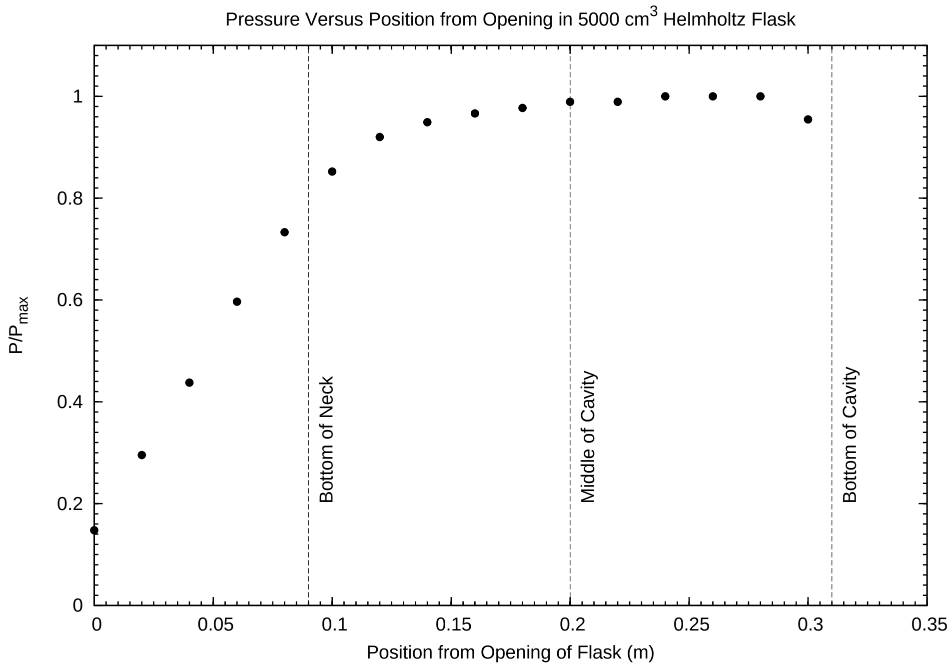


Figure 16: Pressure at various positions in side of a 5000 cm^3 Helmholtz flask.

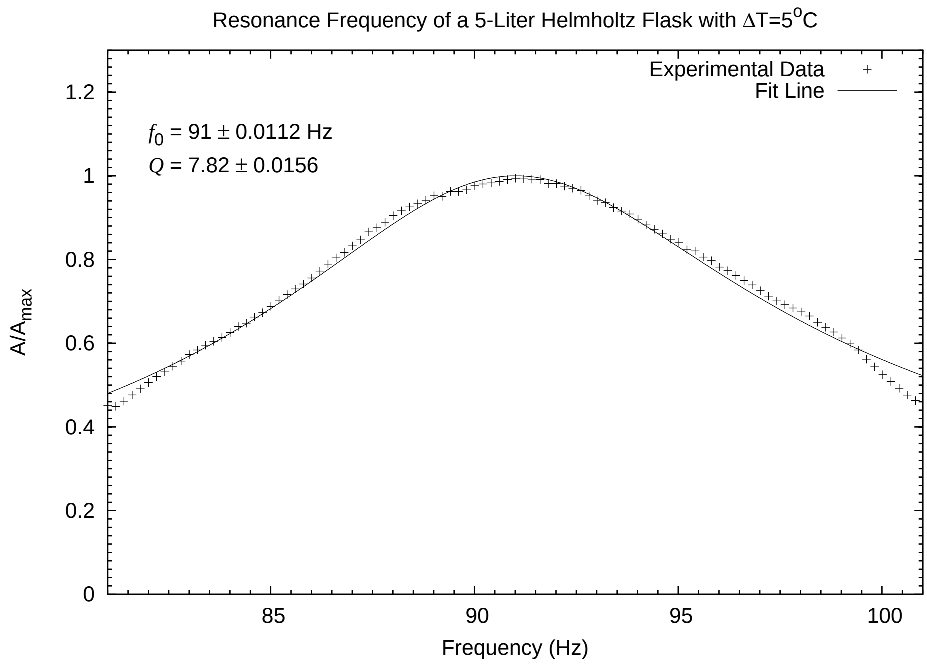


Figure 17: Resonance curve for a 5000 cm^3 Helmholtz flask with a 324 psi ceramic stack placed into its lower neck.

6.3 Resonance Curve for Helmholtz Flask with Ceramic Stack

After inserting the stack into the lower position of the Helmholtz flask's neck, the system was once again swept through a range of frequencies with no temperature difference applied to the stack. Figure 17 represents the resonance peak of this system. The most obvious difference between this graph and the resonance graph for the empty flask (Figure 14) is the broadness of the peak. Inserting the stack into the flask's neck caused the quality factor to significantly decrease, thus indicating an increased resistance. Assuming that the mass of the system (the air oscillating in the neck) has not changed significantly and remembering that the quality factor is inversely proportional to the resistance of the system, one assumes that the introduction of the stack into this system has caused an increase in the system's resistance. This makes sense because the introduction of the stack into the flask causes the surface area within the neck to increase, and the air oscillating in the neck will experience greater friction due to its contact with the pore walls in the stack.

The resonance frequency has decreased, as well, from its value of 97.5 ± 0.003 Hz for the empty flask to 91.0 ± 0.011 Hz. From Equation 10 for a Helmholtz flask's resonance frequency, one sees that both the effective length and the volume of the flask has not changed; however, the insertion of the stack has decreased the neck's cross-sectional area thus decreasing the system's resonance frequency.

6.4 Temperature Difference across Ceramic Stack

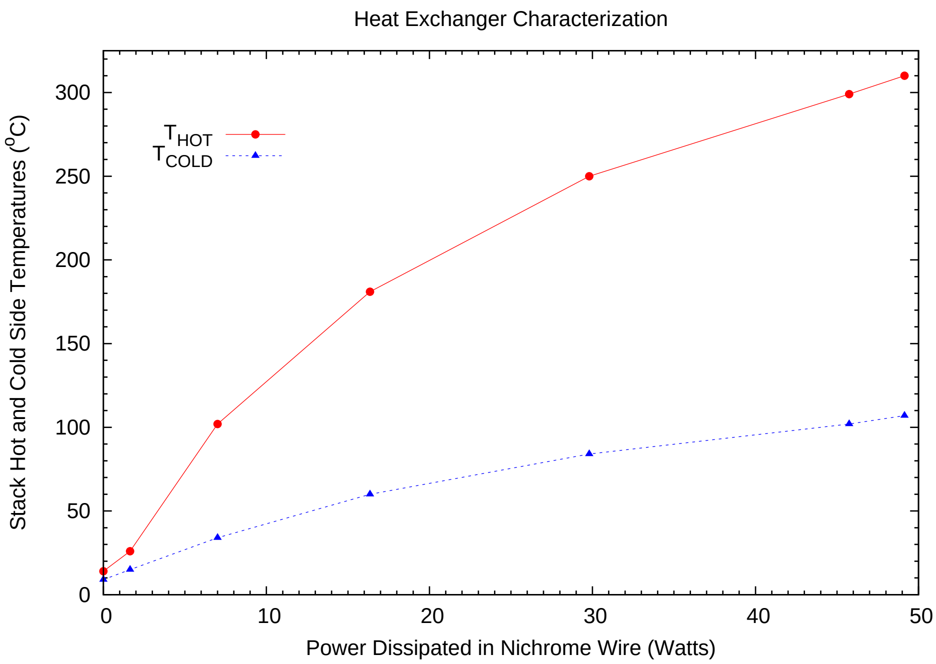


Figure 18: Temperature difference across the substrate as a function of the power dissipated in the nichrome wire. The lines connecting the data points are not of empirical value but serve as a guide for the eye.

Figure 18 shows the temperatures for both sides of the stack with the hot and cold heat exchangers versus the power dissipated by the nichrome wire. From this graph one notices that the cold heat exchanger wasn't able to provide the cool side of the stack with a constant temperature. After increasing the power for each data set, the temperatures were allowed to equilibriate, and the flask was driven through a range of frequencies for each temperature difference. For each set of data taken, an increase in both the resonance frequency and the quality factor was observed. The data for these experiments are displayed in Appendix D.

6.5 Resonance Curve at 203 °C Temperature Difference across Stack

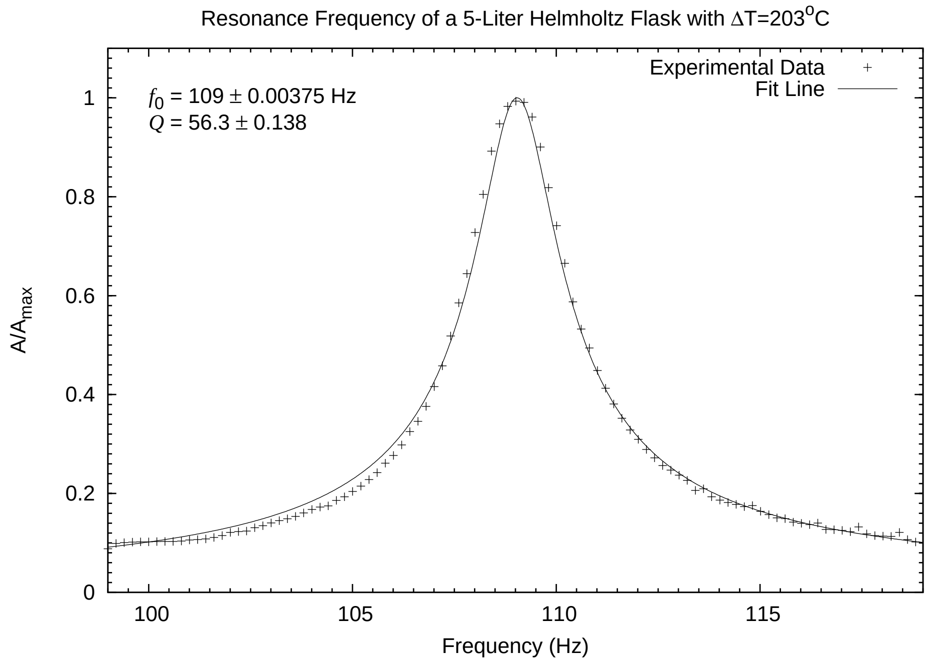


Figure 19: Resonance curve for a temperature difference of 203°C across the stack.

At a temperature difference of 203 °C, the system was just below onset. A resonance curve for this system is shown in Figure 19. The resonance frequency has increased from 91.0 ± 0.011 Hz at a $\Delta T = 5\,°C$ to 109.0 ± 0.004 Hz at a $\Delta T = 203\,°C$, and the quality factor has increased from 7.82 ± 0.016 to 56.5 ± 0.138, indicating a decrease in the total mechanical resistance of the system.

7 Analysis

As the temperature difference across the stack increased, the resonance frequency of the system increased from 91.0 $\pm$ 0.011 Hz at $\Delta T = 5\,^{\circ}$C to a 109.0 $\pm$ 0.004 Hz at $\Delta T = 203\,^{\circ}$C. Referring to the definition for the resonance frequency of a Helmholtz resonator (Equation 10), one notices its dependence upon the speed of sound and the dimensions of the flask. Of course, after the stack is inserted into the flask's neck, the dimensions of the flask are not altered for the remainder of the experiment; therefore, one automatically assumes that an increase in the speed of sound is responsible for the increase in the resonance frequency. The speed of sound is dependent upon temperature (Equation 32), and as more heat is introduced into the system, the temperature of the air within the flask increased causing an increase in the speed of sound. This increase in the speed of sound caused an increase in the flask's resonance frequency.

As the temperature difference across the stack increased, the quality factor is shown to increase, as well. The quality factor increases from 7.82 $\pm$ 0.016 at $\Delta T = 5\,^{\circ}$C to a 56.3 $\pm$ 0.138 at $\Delta T = 203\,^{\circ}$C. The quality factor is defined by Equation 2, where it is shown to be proportional to the angular resonance frequency and the mass of air in the neck and inversely proportional to the resistance in the system. The mass of air within the neck has not changed significantly as the temperature difference across the stack increases, and although the resonance frequency does increase with increasing temperature difference, this increase is not significant enough to cause the observed increase in the quality factor. Therefore, this increase in the quality factor must be associated with a decrease in the system's total mechanical resistance.

Because the quality factor is inversely proportional to the total mechanical resistance in the system, it is more enlightening to study how the resistance of the system is affected as the inverse quality factor changes due to the temperature difference across the stack. Figure 20 shows the inverse quality factor as a function of the temperature difference across the stack. The linear fit of this data was initially used in the experiment as a means to predict the temperature difference at which onset would occur. For this thermoacoustic engine onset occured at $\Delta T = 207\,^{\circ}\mathrm{C}$. At onset the net work of the gas parcels within the stack has become great enough to overcome the resistance caused by the friction due to the motion of the parcels against the walls of the pores. Because the resistance of the system has essentially been eliminated, the system produces sound. Therefore, the thermoacoustic engine at this point may be considered an ideal heat engine.

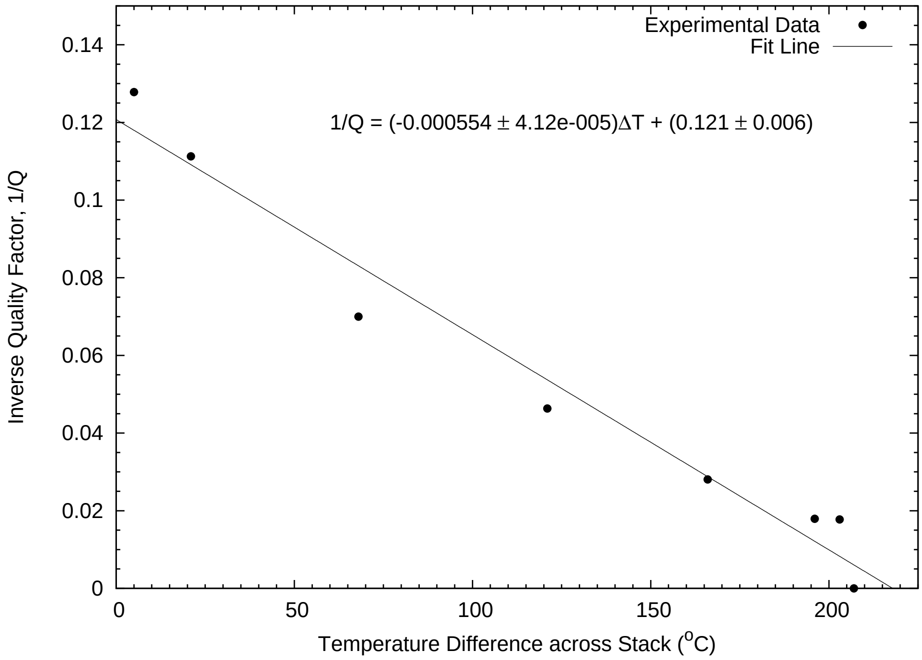


Figure 20: Inverse quality factor versus the temperature difference across the stack.

8 Conclusion

Thermoacoustics is the field of study that strives to invent new technologies that exploit acoustically-driven systems in order to perform work or transfer heat. These engines and refrigerators are preferable to their predecessors in that they are mechanically simple and do not introduce harmful agents into the environment. However, thermoacoustic engines also offer a unique alternative to introducing physical concepts in educational settings. Traditionally, resonance, a physical concept that can have dire consequences regarding the integrity of stuctures and equipment, has been taught in educational settings via mass-spring systems. By presenting this concept through the use of a Helmholtz thermoacoustic engine, students will have the opportunity to see that resonance for a system can be achieved via other physical methods.

Appendix A: Derivation for the Driven, Damped Spring-mass System

Newton's second law for a driven, damped spring-mass system with a sinusoidal driving force is:

$$f(t)e^{i\omega t} - Rv - kx = ma \tag{12}$$

$$m\frac{d^2x}{dt^2} + R\frac{dx}{dt} + kx = fe^{i\omega t} \tag{13}$$

In this equation x is the displacement of the mass, m is the mass, R is the resistance, and k is the spring constant. The term on the right-hand-side of the equation represents the external sinusoidal force driving the system. Assuming a particular solution to this differential equation of the form:

$$x_p(t) = xe^{i\omega t} \tag{14}$$

Inserting this solution into the differential equation yields:

$$(-m(\omega)^2 + Ri(\omega) + k)xe^{i(\omega)t} = fe^{i(\omega)t} \tag{15}$$

$$x = \frac{f}{-m(2\omega)^2 + Ri(\omega) + k} \tag{16}$$

In order to eliminate the imaginary parts of the displacement, x is multiplied by its complex conjugate.

$$
\begin{aligned}
x^2 = x^*x &= \left(\frac{f}{-m(\omega)^2 - Ri(\omega) + k}\right)\left(\frac{f}{-m(\omega)^2 + Ri(\omega) + k}\right) \\
x^2 &= \frac{(f)^2}{(\nu_0^2 - \nu^2)^2 + \frac{R^2\nu^2}{4\pi^2 m^2}} \\
x^2 &= \frac{(f)^2}{\sqrt{(\nu_0^2 - \nu^2)^2 + \frac{R^2\nu^2}{4\pi^2 m^2}}} \\
|x| &= \frac{f}{\nu_0^2\sqrt{(1 - \frac{\nu^2}{\nu_0^2})^2 + \frac{R^2\nu^2}{4\pi^2\nu_0^4 m^2}}} \\
|x| &= \frac{f}{\nu_0^2\sqrt{(\frac{1}{Q^2})(\frac{\nu^2}{\nu_0^2}) + (1 - \frac{\nu^2}{\nu_0^2})^2}} \\
|x| &= \frac{f}{\frac{\nu_0\nu}{Q}\sqrt{1 + (\frac{\nu_0}{\nu})^2 Q^2[1 - (\frac{\nu}{\nu_0})^2]^2}} \qquad (17)
\end{aligned}
$$

Equation 17 is the general solution for a simple harmonic oscillator. The resonance frequency of the driven-damped spring-mass system, ν_o, is given by

$$\nu_o = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (18)$$

The quality factor for this system is:

$$Q = \frac{\omega_o m}{R} \qquad (19)$$

When the system is at resonance, $\nu = \nu_o$, and Equation 17 becomes:

$$x_{max} = \frac{\nu Q}{4\pi^2\nu_o} \qquad (20)$$

Inserting this value for x_{max} into Equation 17 yields:

$$\frac{x}{x_{max}} = \frac{1}{\sqrt{1 + \left(\frac{\nu_0}{\nu}\right)^2 Q^2 \left[1 - \left(\frac{\nu}{\nu_0}\right)^2\right]^2}} \tag{21}$$

Appendix B: Derivation of Euler's Equation

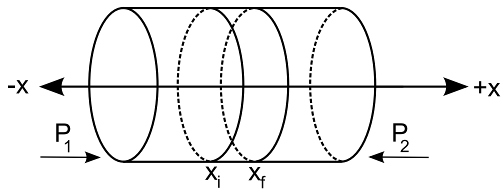


Figure 21: Pipe with pressures P_1 and P_2 at either openings.

Consider a pipe as shown in Figure 21. Assume that the pressure P_2 is greater than the P_1; therefore, the change in pressure, ΔP, is negative. Because pressure is defined as the force per unit area, A, Newton's Second Law can be written in terms of the pressure.

$$\begin{aligned} \Sigma F &= ma \\ -A\Delta P &= ma \end{aligned} \tag{22}$$

Because acceleration is the first derivative of the velocity, Equation 22 can be written as:

$$-A\Delta P = m\frac{dv}{dt} \tag{23}$$

Considering a pressure change over a small volume in the pipe, $V = A(x_f - x_i) = A\Delta x$, and dividing both sides of Equation 23 by this volume yields:

$$\begin{aligned} -\frac{A\Delta P}{A\Delta x} &= \frac{m}{A\Delta x}\frac{dv}{dt} \\ -\frac{\Delta P}{\Delta x} &= \frac{m}{A\Delta x}\frac{dv}{dt} \end{aligned} \tag{24}$$

The density of the air for ambient conditions, ρ_o, is defined as the mass per unit volume of air:

$$\begin{aligned} \rho_o &= \frac{m}{V} \\ \rho_o &= \frac{m}{A\Delta x} \end{aligned} \tag{25}$$

Inserting this value for the ambient air density into Equation 24 gives:

$$-\frac{\Delta P}{\Delta x} = \rho_o \frac{dv}{dt} \tag{26}$$

Due to the sinusoidal nature of the pressure and velocity, the pressure and velocity are represented by $P = P_o(x)e^{\omega_o t}$ and $v = u_{AC}(x)e^{\omega_o t}$, respectively. Inserting this value for pressure and the derivative of the velocity with respect to time into Equation 26 yields:

$$\begin{aligned} \frac{-\Delta P_o(x)e^{i\omega_o t}}{\Delta x} &= (i\omega_o\rho_o)(u_{AC}(x)e^{i\omega_o t}) \\ \frac{-\Delta P_o(x)}{\Delta x} &= (i\omega_o\rho_o)(u_{AC}(x)) \\ u_{AC} &= \left| \frac{1}{\omega_o\rho_o} \frac{\Delta P_o}{\Delta x} \right| \end{aligned} \tag{27}$$

Appendix C: Derivation for the Acoustical Driven, Damped System

The equation for the ratio of the displacement to the maximum displacement at resonance for the air driven inside of the Helmholtz flask is:

$$\frac{x}{x_{max}} = \frac{1}{\sqrt{1 + \left(\frac{\nu_0}{\nu}\right)^2 Q^2 \left[1 - \left(\frac{\nu}{\nu_0}\right)^2\right]^2}} \tag{28}$$

The displacement of the mass within the Helmholtz flask is not easily measurable. However, it is possible to measure the pressure changes occuring in the flask. Therefore, it is necessary to express equation 28 in terms of a ratio of pressures rather than a ratio of displacements.

Sound is an adiabatic process, which means that the compression and expansion of gas occurs quickly such that there is no transfer of heat. The adiabatic Equation of state is

$$\frac{P}{P_o} = \left(\frac{V_o}{V}\right)^{\gamma} \tag{29}$$

where P is the pressure, P_o is the ambient pressure, V is the volume, and V_o is the ambient volume.[3] Because this experiment focuses upon small changes in pressure and velocity, P and V can be written as $P = P_o + \Delta P$ and $V = V_o + \Delta V$, where P_o and V_o are the initial values for pressure and velocity and ΔP and ΔV are the small changes in pressure and volume. Inserting these values for pressure and velocity into the adiabatic equation of state and simplifying the equation yields

$$\frac{\Delta P}{P_o} = -\gamma \frac{\Delta V}{V_o} \tag{30}$$

The change in the flask's volume is $\Delta V = -Sx$. If the mass moves down into the neck, the volume decreases whereas the volume increases when the mass moves towards the opening of the flask. Substituting this value into Equation 30 gives

$$\frac{\Delta P}{P_o} = \frac{\gamma S x}{V_o} \tag{31}$$

The speed of sound, c, is represented by the equation:

$$c^2 = \gamma R' T \tag{32}$$

In order to obtain a value for the ratio of specific heats, γ, we solve Equation 32 for γ and replace R' with $\frac{P_o}{\rho_o T}$, which comes from the ideal gas law. ρ_o is simply the ambient density of the air within the flask. Inserting our definition for γ into Equation 31 and solving for the pressure yields

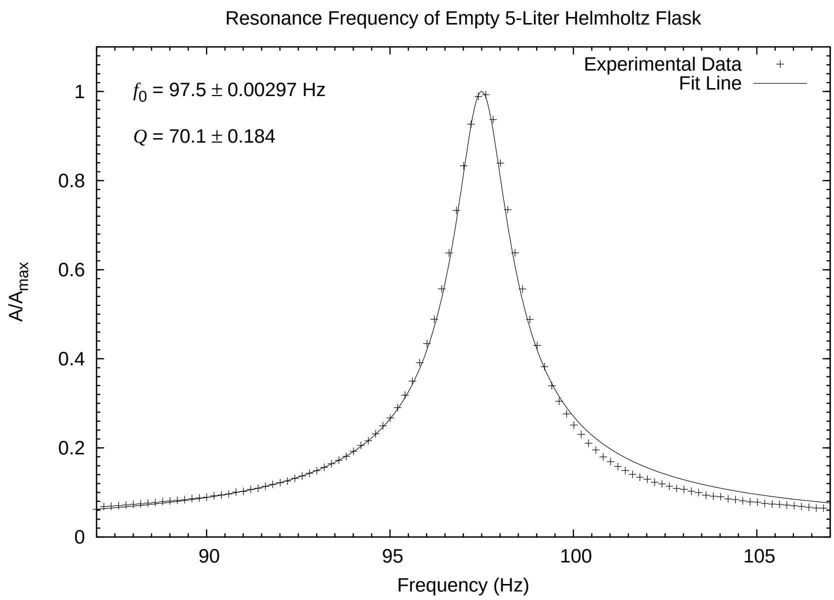
$$P = \frac{\rho_o c^2 S}{V_o} x \tag{33}$$

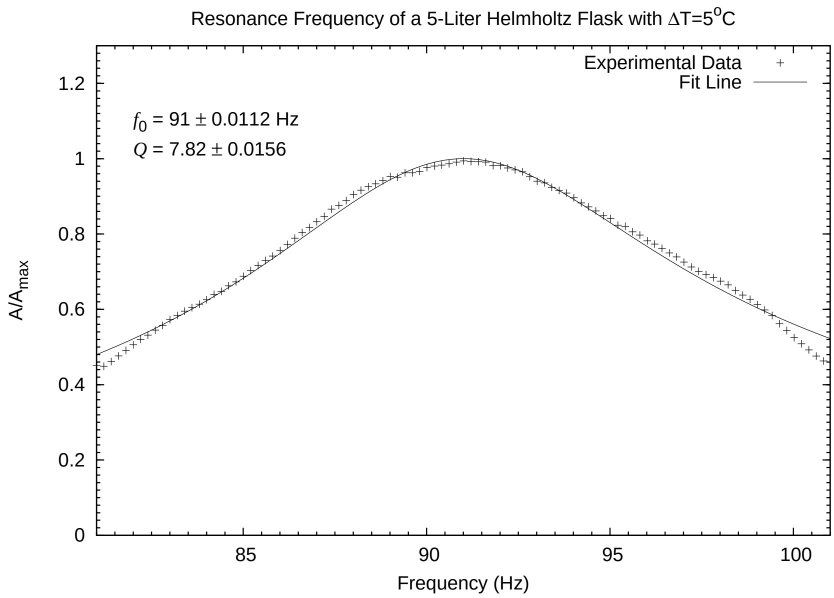
This equation shows the acoustic pressure as a function of the acoustic displacement. Solving this equation for the displacement and inserting this definition for the displacement into Equation 28 gives

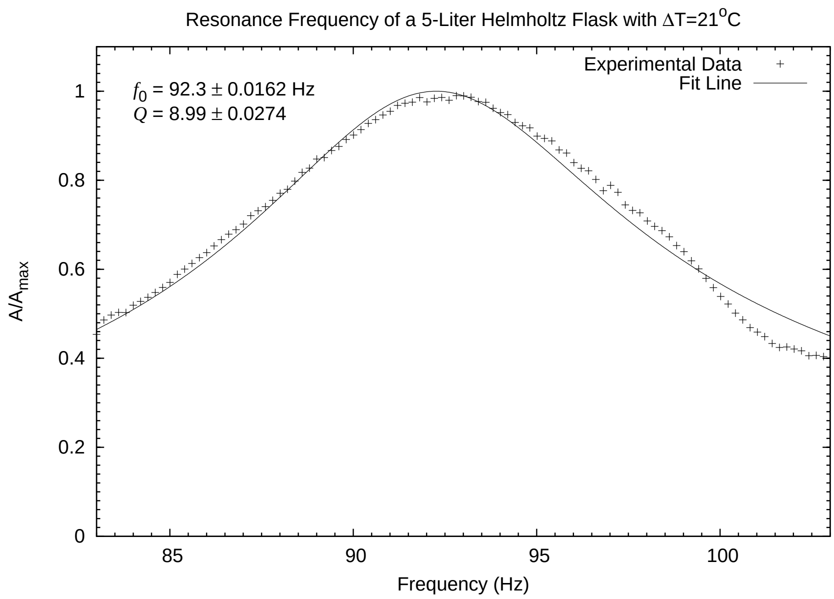
$$\frac{A}{A_{max}} = \frac{1}{\sqrt{1 + \left(\frac{\nu_0}{\nu}\right)^2 Q^2 \left[1 - \left(\frac{\nu}{\nu_0}\right)^2\right]^2}} \tag{34}$$

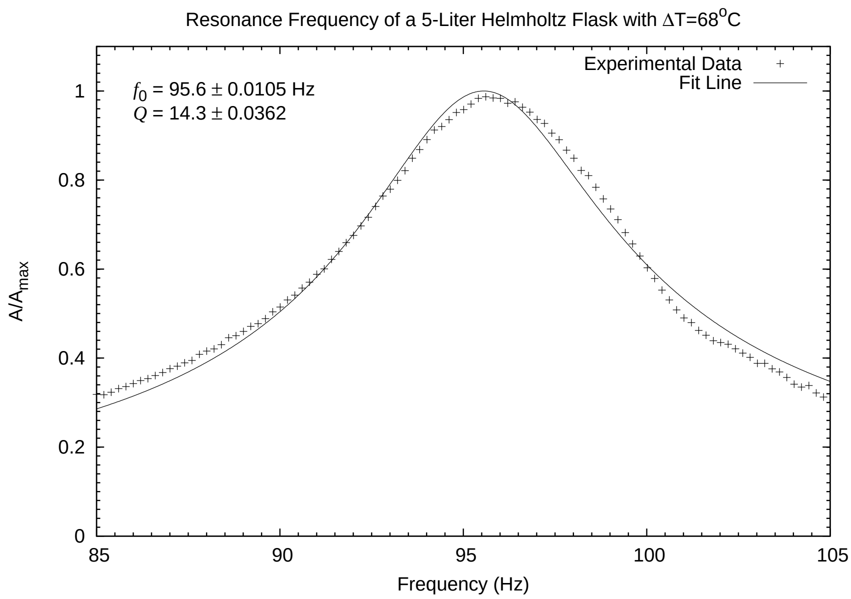
Here, A is the amplitude of the pressure and A_{max} is the amplitude of the pressure at resonance.[9]

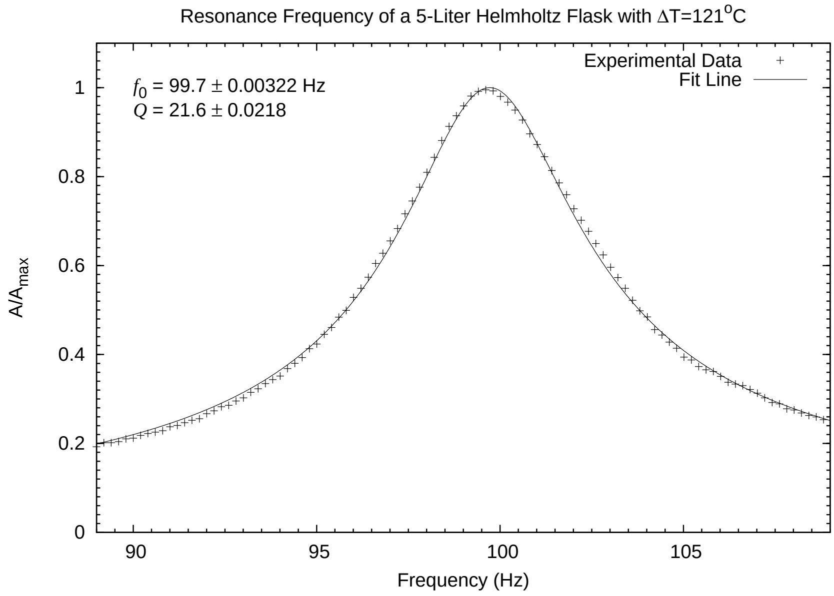
Appendix D: Resonance Curves for Increasing Temperature Differences

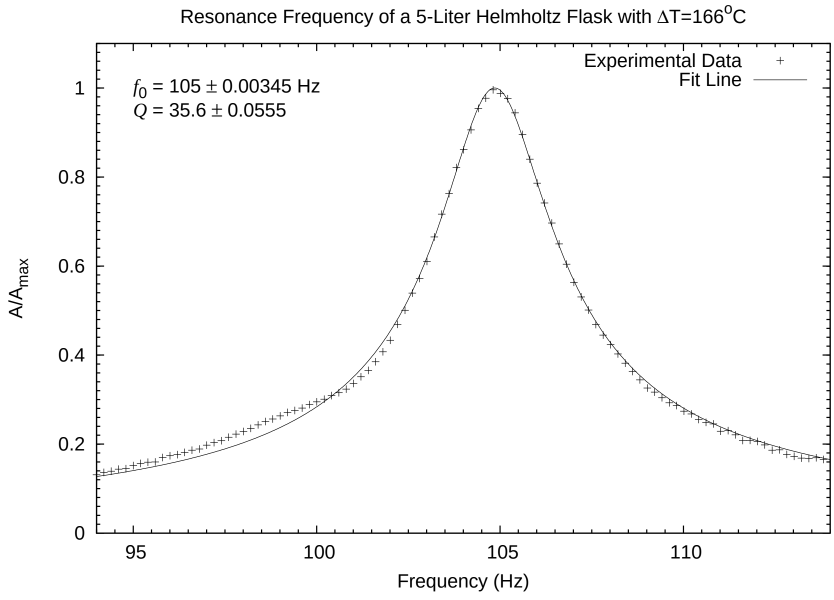


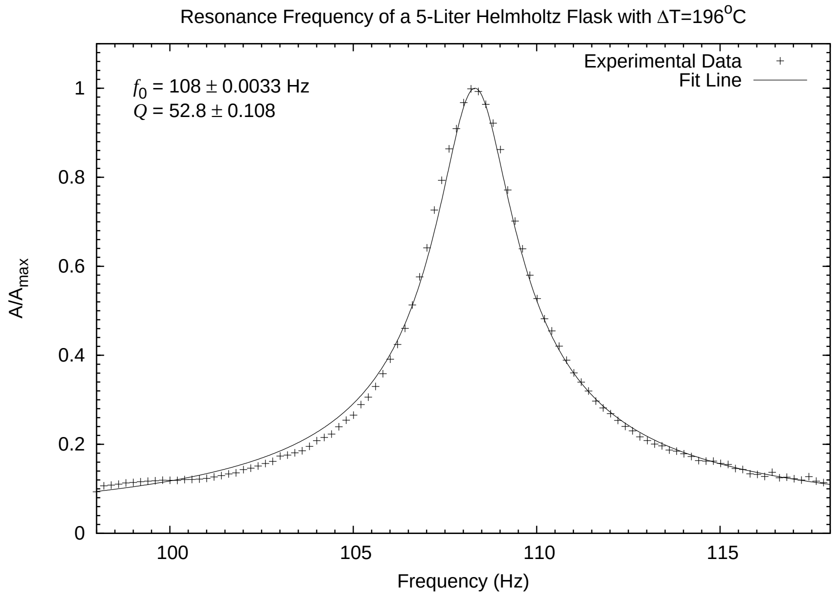



Resonance Frequency of a 5-Liter Helmholtz Flask with ΔT=21°C
Experimental Data
Fit Line
f_0 = 92.3 ± 0.0162 Hz
Q = 8.99 ± 0.0274
A/A$_{max}$
Frequency (Hz)
0
0.2
0.4
0.6
0.8
1
85
90
95
100

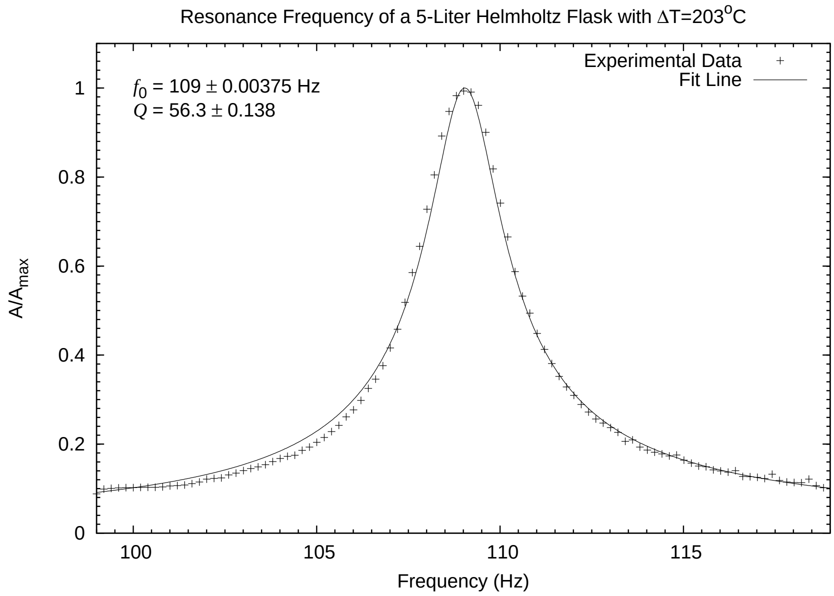

Resonance Frequency of a 5-Liter Helmholtz Flask with ΔT=68°C
Experimental Data
Fit Line
f_0 = 95.6 ± 0.0105 Hz
Q = 14.3 ± 0.0362
A/A$_{max}$
Frequency (Hz)
0
0.2
0.4
0.6
0.8
1
85
90
95
100
105


Resonance Frequency of a 5-Liter Helmholtz Flask with ΔT=121°C
Experimental Data
Fit Line
f_0 = 99.7 ± 0.00322 Hz
Q = 21.6 ± 0.0218
A/A$_{max}$
0
0.2
0.4
0.6
0.8
1
90
95
100
105
Frequency (Hz)


Resonance Frequency of a 5-Liter Helmholtz Flask with ΔT=166°C
Experimental Data
Fit Line
f_0 = 105 ± 0.00345 Hz
Q = 35.6 ± 0.0555
A/A$_{max}$
0
0.2
0.4
0.6
0.8
1
95
100
105
110
Frequency (Hz)



Resonance Frequency of a 5-Liter Helmholtz Flask with ΔT=196°C
f_0 = 108 ± 0.0033 Hz
Q = 52.8 ± 0.108
Experimental Data
Fit Line
A/A$_{max}$
Frequency (Hz)



Resonance Frequency of a 5-Liter Helmholtz Flask with ΔT=203°C
f_0 = 109 ± 0.00375 Hz
Q = 56.3 ± 0.138
Experimental Data
Fit Line
A/A$_{max}$
Frequency (Hz)

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