BESSEL'S FLAMES OF GLORY

Shane McNew, Faculty Mentor: Dr. William V. Slaton, University of Central Arkansas

Department of Physics and Astronomy

ABSTRACT

The Ruben's Tube is a popular physics demonstration that dramatically illustrates one-dimensional standing waves. The demo consists of a long tube closed on one end and the other driven with speaker. The tube is filled with natural gas that exits through small evenly spaced holes along the top of the tube. The gas exiting the holes is lit and the heights and color of the flames are an indication of the speaker-driven standing wave that appears in the tube. Regions where the standing wave acoustic velocity is large are illustrated by tall yellow flames, whereas regions with low acoustic velocity have short bright blue flames. This variation in flame height is due to the Bernoulli effect. This research project explored the theoretical and experimental design, construction and testing of two-dimensional flame pans: square and circular. The square geometry should support standing waves similar to the Ruben's tube in both directions and so the flame patterns will be lines of yellow flames in a grid pattern. The circular geometry should support standing waves that have a radial and polar dependence with corresponding flame patterns. Theoretical predictions of the resonance modes in the two-dimensional flame pans will be compared to photographs of the flame pans in operation.

THEORY

The flame heights are caused by a phenomena known as the Bernoulli effect. The Bernoulli effect says that the pressure will be a minimum where the acoustic velocity is a maximum. Where the pressure is a minimum is where the yellow flames will be visualized. To find where the yellow flames will occur, the equation of continuity

$$\frac{\partial \rho}{\partial t} = -(\vec{\nabla} \cdot \rho \vec{v})$$

and Euler's Equation

$$-\vec{\nabla} P = \rho \frac{\partial \vec{v}}{\partial t}$$

These equations can be combined using the ideal gas law

$$PV = nRT$$

The ideal gas law can be rewritten as

$$P = \rho R^* T$$

Where ρ = nm/V is the density of the gas, and

$$R^* = \frac{R}{m}$$

Using the ideal gas law to combine the continuity and Euler equations gives the wave equation

$$\nabla^2 P = \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2}$$

Where c is the speed of sound. We assume the pressure is separable into a time-dependent function and a spatially-dependent function. We also ignore solutions in the z-direction and assume the time dependent portion goes as

$$T(t) = Cos(\omega t)$$

Where ω = ck.

The spatial-dependent equation is known as the Helmholtz equation

$$\nabla^2 P = -k^2 P$$

Rectangular Coordinates:

The rectangular coordinates solution to the Helmholtz equation is

$$P(x,y) = P_0 Cos(k_x x) Cos(k_y y)$$

Where $k_x^2 + k_y^2 = k^2$, these values come from the boundary conditions we impose. That is the velocity must be zero at the boundaries.

$$\frac{\partial P}{\partial x} = 0 \ \ at \ \ x = 0 \ \ and \ \ x = L$$

$$\frac{\partial P}{\partial y} = 0 \ \ at \ \ y = 0 \ \ and \ \ y = L$$

This leads to $k_x L = m\pi$ and $k_y L = n\pi$, where L is the length of one of the edges of the square pan and m,n are integers greater than or equal to zero.
The frequencies are found using

$$f_{exp} = \frac{c\sqrt{m^2 + n^2}}{2L}$$

Cylindrical Coordinates:

The cylindrical coordinates solution of the Helmholtz equation is of the form of Bessel's solutions.

$$P(r,\theta) = P_0 J_m(kr) Cos(m\theta)$$

The k value is found using the same boundary condition only in cylindrical coordinates it is of the form

$$\frac{\partial}{\partial r} J_m(kr) = 0 \ \ at \ \ r = R$$

Some of the zeroes (of the form $j_{m,n} = k_n R$) are

$$j_{1,1} = 1.84118$$
$$j_{2,1} = 3.05424$$
$$j_{0,1} = 3.83171$$
$$j_{3,1} = 4.20119$$
$$j_{2,2} = 6.70613$$

The frequencies are found using

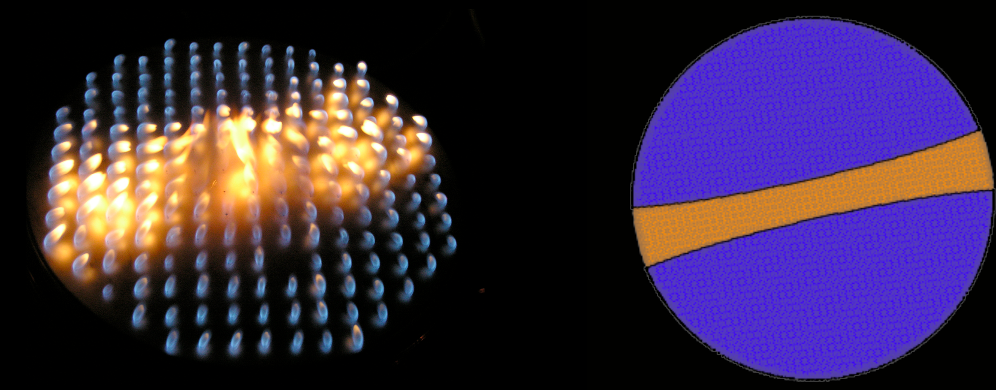
$$f_{m,n} = \frac{c}{2\pi} \frac{j_{m,n}}{R}$$

MODELLING

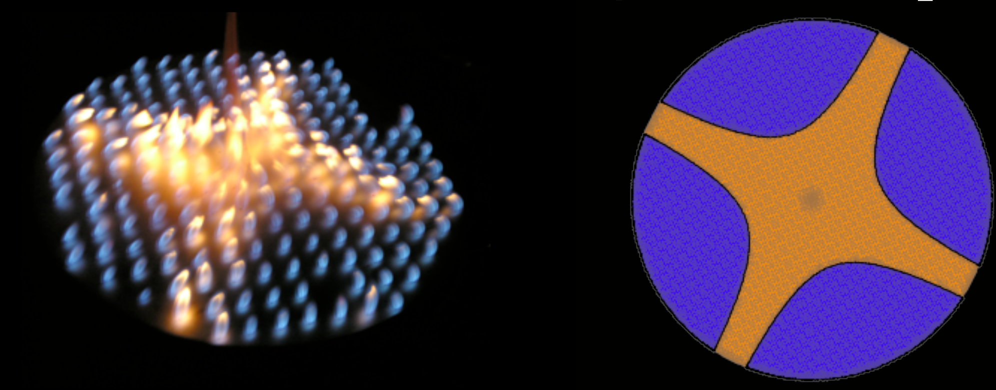
The pressures P(x,y) and P(r,θ) were plotted using Mathematica's ContourPlot function. The orange region on the plots represents where ∇P is large hence, this is where the acoustic velocity is very large. Outside of the region, the blue region, is where ∇P is small, so the acoustic velocity is small. Where the velocity is small, the gas pressure is high and the flames burn a hot blue. Where the velocity is large, the gas pressure is low the flames burn a cooler yellow.

The speed of sound in the gas was calculated to be 519 m/s. The square pan was 0.405 m in length and the radius of the circular pan was 0.2025 m.

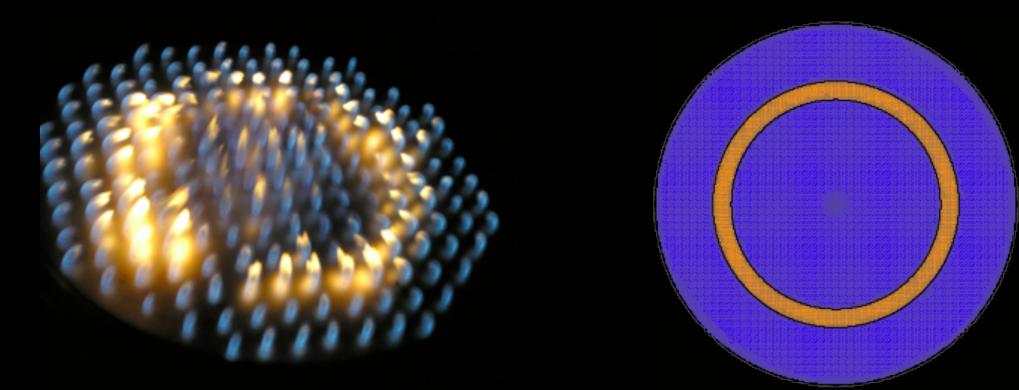
CIRCULAR FLAME PAN



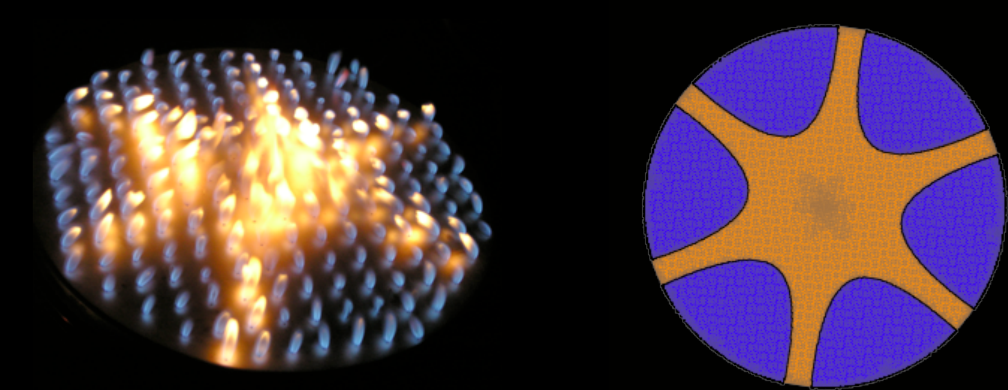
m = 1, n = 1 f_{obs} = 880 Hz f_{exp} = 751 Hz $\%_{rel\_error}$ = 17



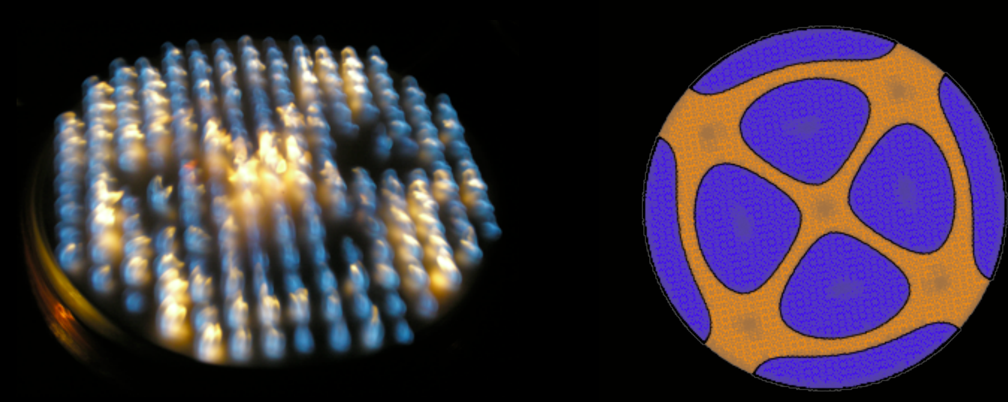
m = 2, n = 1 f_{obs} = 1280 Hz f_{exp} = 1245 Hz $\%_{rel\_error}$ = 2.8



m = 0, n = 1 f_{obs} = 1610 Hz f_{exp} = 1562 Hz $\%_{rel\_error}$ = 3.1

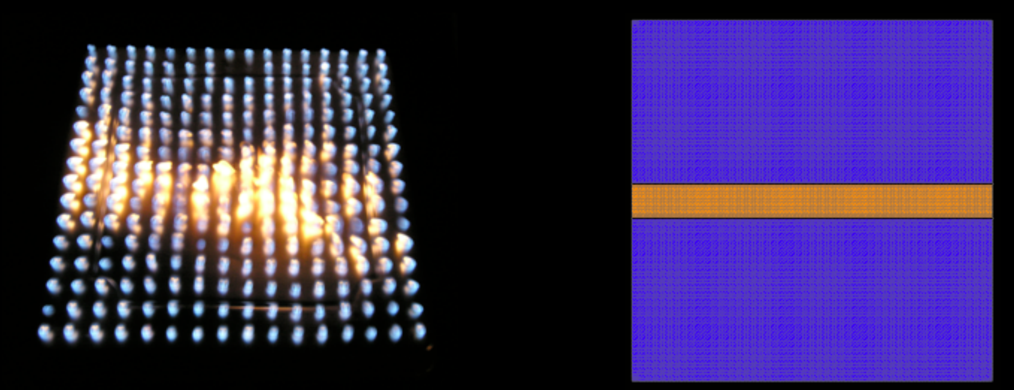


m = 3, n = 1 f_{obs} = 1740 Hz f_{exp} = 1713 Hz $\%_{rel\_error}$ = 1.6

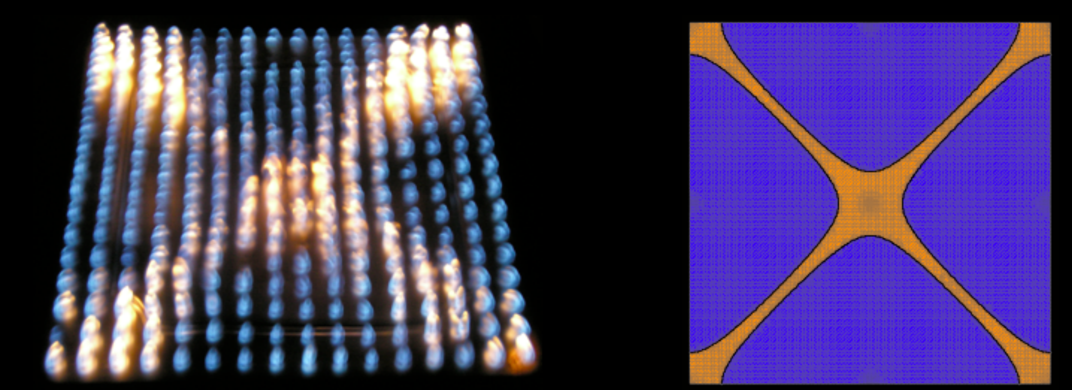


m = 2, n = 2 f_{obs} = 2720 Hz f_{exp} = 2735 Hz $\%_{rel\_error}$ = 0.55

SQUARE FLAME PAN



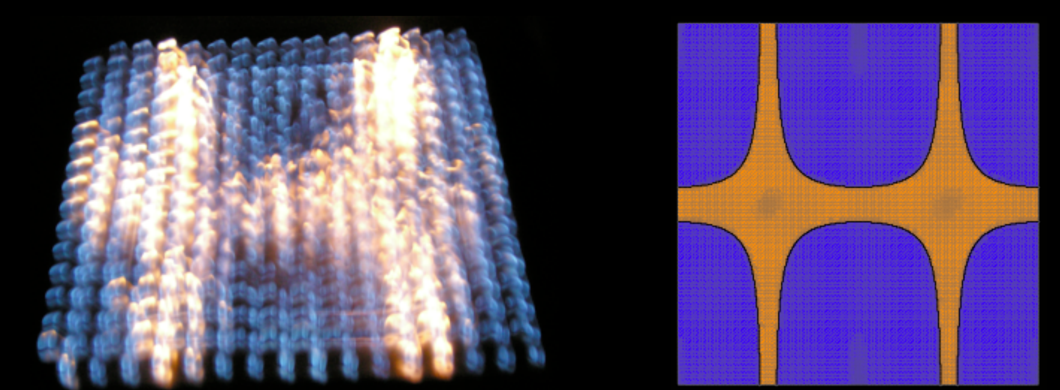
m = 0, n = 1 f_{obs} = 705 Hz f_{exp} = 641 Hz $\%_{rel\_error}$ = 10



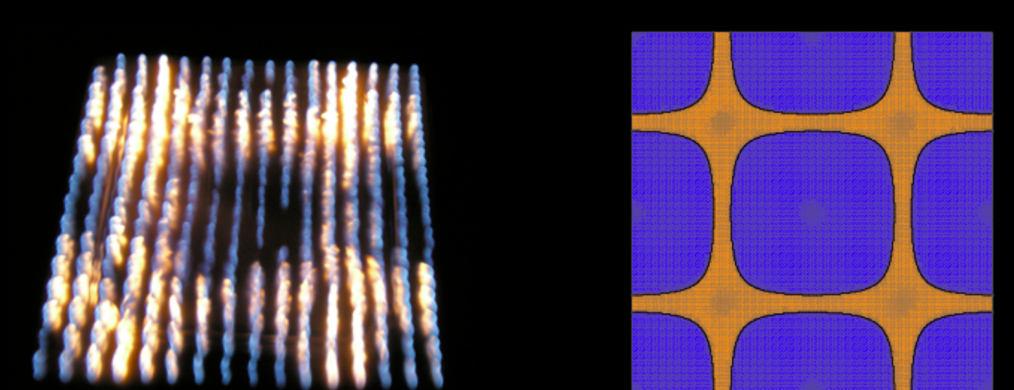
m = 2, n = 0

f_{obs} = 1320 Hz f_{exp} = 1281 Hz $\%_{rel\_error}$ = 3.0

m = 0, n = 2



m = 2, n = 1 f_{obs} = 1500 Hz f_{exp} = 1433 Hz $\%_{rel\_error}$ = 4.7



m = 2, n = 2 f_{obs} = 1880 Hz f_{exp} = 1812 Hz $\%_{rel\_error}$ = 3.8

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