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# Determining the Coefficient of Discharge for a Draining Container

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The flow of fluids through open containers is a topic studied frequently in introductory physics classes. A fluid mechanics class delves deeper into the topic of fluid flow through open containers with holes or barriers. The flow of a fluid jet out of a sharp-edged orifice rarely has the same area as the orifice due to a fluid flow phenomenon known as the vena contracta. The area of a fluid jet out of an orifice is related to the actual area of the orifice by a value known as the coefficient of discharge, $C_d$. The purpose of this paper is to develop a mathematical model for a draining open container and an experimental method that will efficiently determine the coefficient of discharge for such a system. Prior work in physics education literature has developed a method for measuring the flow of a fluid out of an orifice using ultrasonic motion detectors. In this paper we present data that show our method can be used to find the coefficient of discharge within the expected literature values for sharp-edged and rounded orifices.

## Introduction

A desire for a calculus-based fluids lab for undergraduate physics majors prompted this investigation into fluid draining from an orifice at the bottom of a cylindrical open-top container. A review of the literature on the topic proved to be fascinating. In 1908, Judd and King[1] chronicled an experiment to measure the volumetric flow rate through a sharp-edged orifice as a function of water depth. Of interest was the discharge coefficient, $C_d$, defined as the ratio of the actual volume discharge rate divided by that predicted by Torricelli's law. They used an ingenious micrometer caliper system to measure the shape of the water jet issuing from the orifice. This information quantitatively showed that the jet area is smaller than the orifice area for a sharp-edged orifice [see Fig. 1(b)]. This phenomenon is known as the vena contracta. During the course of their experiment, Judd and King observed small changes in the discharge coefficient as a function of water depth, which is related to the water exit velocity. In 1933, Tuve and Sprenkle[2] investigated the vena contracta phenomenon in an orifice plate placed inside a pipe with viscous fluid flow. Their experiment, while not exactly the same as Judd and King, showed that the discharge coefficient is a function of the flow's exit velocity through the orifice, especially for Reynold's numbers between 1 and 10,000. In 1940, Medaugh and Johnson[3] return to Judd and King's experiment and measured discharge coefficients for a broader range in fluid head height, demonstrating that the discharge coefficient depends on the fluid's exit velocity. Interest in jetting from orifices returned to the forefront with the invention of ink-jet printers in the 1980s. Leinard and Leinard[4] determined that "surface tension does not retard a liquid jet unless it completely stops it." In 2003, Libii[5] demonstrated that fluid draining from a cylindrical tank marked with a measuring tape for visual data recording with a stopwatch can be incorporated into the undergraduate laboratory experience. In 2005, Saleta et al.[6] and Escamilla[7] also demonstrated an undergraduate lab exploring this topic. To visualize fluid flow they use a digital photograph of the jet exiting from the orifice in the side of a cylinder. Graph paper in the background of the photograph allows the student to determine the exit velocity of the jet using kinematics. Most recently, Guerra et al.[8-9] have demonstrated how to use ultrasonic motion detectors to rapidly measure the velocity of the surface of the fluid in an open-top container as it drains. This velocity can be related to the fluid's velocity through the orifice via the equation of continuity.

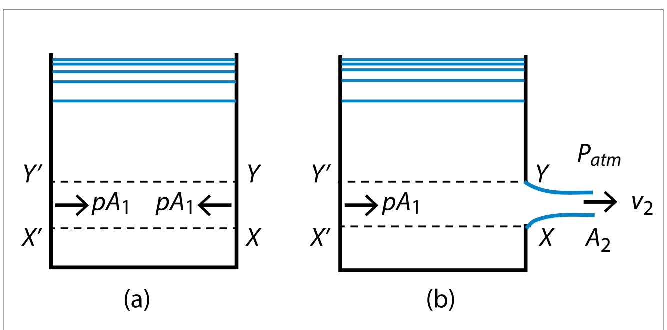


**Fig. 1. Illustrating presence of vena contracta in fluid jet issuing from side of an open-topped fluid-filled tank.**

Lamb's *Hydrodynamics*[10] text and Milne-Thompson's *Theoretical Hydrodynamics*[11] text explore fluid motion from a high mathematical level. However, both texts use a simple physical argument early in the discussion of fluid draining from an open-topped container to argue why the cross-sectional area of the fluid jet issuing from a sharp-edged orifice in the side of an open-topped container must be smaller than the cross-sectional area of the orifice, which we will reproduce here. Figure 1 illustrates a small hole $XY$ in the side of a straight-walled open-topped container that is filled with fluid, maintained at a fixed level to simplify the discussion. The atmospheric pressure outside the container is $P_{atm}$ and $v_2$ is the speed of fluid exiting the hole as measured at the vena contracta. The hydrostatic pressure at the depth of the hole is $p$. When the hole is closed, as illustrated in Fig. 1(a), the net force of the container's walls on the fluid is zero. However, when the hole is opened, as illustrated in Fig. 1(b), the force at $XY$ becomes $p_{atm}A_1$. Hence, if we assume that the hydrostatic pressure, $p$, is unaltered except at the orifice, then the net force on the fluid at the orifice is: $(p - p_{atm})A_1$, which must equal the time rate of change of the momentum,

$$(p - p_{\text{atm}})A_1 = \frac{d}{dt}(mv_2). \tag{1}$$

If the fluid level in the tank is held fixed, then the fluid's exit velocity at the vena contracta is a constant, $v_2$. The rate at which the mass of fluid is exiting the container may be written in terms of the fluid's density, the jet's cross-sectional area, and speed as $dm/dt = \rho A_2 v_2$. Thus, Newton's second law for the fluid is

$$(p - p_{\text{atm}})A_1 = \rho A_2 v_2^2, \tag{2}$$

which may be combined with Bernoulli's equation for a streamline that flows through the center of the orifice,

$$p + \frac{1}{2}\rho v_1^2 = p_{\text{atm}} + \frac{1}{2}\rho v_2^2, \tag{3}$$

where the left-hand side of the equation refers to the near-stationary fluid in the container and the right-hand side refers to the fluid in the jet at the vena contracta. Since $v_1$ is small we can ignore $v_1^2$ on the left-hand side. Combining Eqs. (2) and (3) to eliminate $(p - p_{atm})$ yields an expression relating the cross-sectional area of the orifice, $A_1$, and the cross-sectional area of the jet, $A_2$,

$$A_2 = \frac{1}{2}A_1, \tag{4}$$

showing that the coefficient of contraction of the jet is 1/2 for a sharp-edged orifice at this level of approximation. For a round-edged orifice such as a nozzle, the change in cross-sectional area of the jet occurs within the nozzle itself. Another way to explain the existence of the vena contracta without mathematics is to note that the fluid within the container is moving in a radial direction toward the orifice. The fluid maintains this radial component of its velocity upon exiting the container and so the jet narrows as the fluid streamlines become parallel. This curving of the fluid streamlines occurs within the nozzle of a rounded-edge orifice so that the jet emerges with all streamlines parallel. The physical argument presented here is not changed if the orifice is at the bottom of the container.

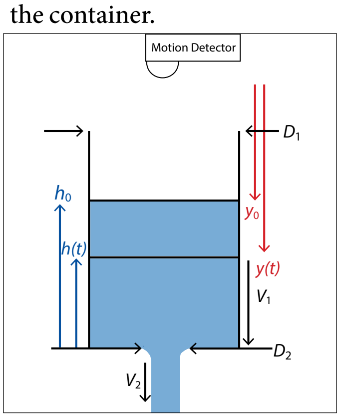


**Fig. 2. Schematic of system with original origin (blue) and changed origin (red).**

## Mathematical model

To begin, we seek to mathematically define the system of interest. Consider an open-topped container where water is flowing out of a given diameter opening in the bottom of a perfect cylinder. The top of the cylinder has a known diameter $D_1$ and area $A_1$, and the hole has a known diameter $D_2$ and known area $A_2$. The water in the cylinder has a certain height $h$. The water is flowing out of the hole at the bottom with a velocity $v_2$ and, consequently, the water level in the top is dropping with a velocity $v_1$ (see Fig. 2).

Starting with the equation of continuity for the fluid, $A_1 v_1 = A_2 v_2$, and Bernoulli's equation,

$$\frac{1}{2}\rho v_1^2 + \rho g h = \frac{1}{2}\rho v_2^2, \tag{5}$$

the following standard expression for the velocity of the fluid within the container may be derived,

$$v_1^2 = \frac{2gh}{A_1^2 / A_2^2 - 1} = Ch, \;\text{ where } C = \frac{2g}{A_1^2 / A_2^2 - 1}, \tag{6}$$

where we have introduced the constant $C$ as indicated. If we had solved for $v_2$ rather than $v_1$, we would have found Torricelli's theorem for the exit velocity of a fluid jet through a hole that is much smaller than the cross-sectional area of the container. Note that Eq. (6) clearly shows that as the fluid drains, $h \to 0$, the velocity of the top surface gets smaller and reaches zero when the container is fully drained, and therefore the slope of the position-versus-time graph for the fluid surface should flatten out when the container is drained. This makes sense physically because as the container drains there is a smaller and smaller pressure difference forcing the fluid out of the hole and, hence, a smaller exit velocity.

Note that $v_1 = -dh/dt$ since $h$ is measured from the bottom of the container. Hence, with this substitution, Eq. (6) is separable and integrates to give

$$h(t) = \left(\sqrt{h_0} - \frac{1}{2}\sqrt{C}(t - t_0)\right)^2. \tag{7}$$

This equation can be used to calculate a theoretical time to drain the container by setting $h(t_{drain}) = 0$ and solving for $t_{drain}$, and then comparing[9] to experimentally measured values.

Previous work[8] relies on a standard ultrasonic motion detector[12] to analyze the draining system. For convenience in our experimental setup, we recast our system to have an origin at the motion detector, which gives the function for the distance of the fluid level from the motion detector.

$$y(t) = y_{\text{f}} - \left[\sqrt{y_{\text{f}} - y_{\text{i}}} - \frac{1}{2}\sqrt{C}(t - t_{\text{i}})\right]^2. \tag{8}$$

The water's initial and final distances from the detector are $y_i$ and $y_f$, and $t_i$ is the initial time when the drain starts. This is the appropriate function for the height of fluid draining from a straight-walled container whose origin is at the position of the motion sensor.

## Experimental setup

For the purpose of this experiment we sought a setup that was similar to the mathematical model. The most cost-effective and accurate setup was to use a straight, open-ended 6-in diameter PVC pipe that is approximately a foot in length. One

**Fig. 3. PVC test caps with holes drilled in the center, grommet and no grommet case shown.**

end of the pipe is cemented into a piece of 1/2-in thick PVC sheeting that has an approximately 4-in-diameter opening coaxial with the 6-in PVC pipe. The opening allows the use of 4-in-diameter PVC test caps (see Fig. 3) with various sized holes drilled in them. A rubber band with width of approximately 1/4-in is placed around the inner edge of the test cap to serve as a seal that keeps the water from leaking out of the system. Test caps make it easy to change the size or properties of the hole being drained and are readily available at home centers and hardware stores. The ease of altering properties of the hole is essential for a teaching lab experiment.

The 6-in PVC pipe is situated in such a way as to be held stable while water is being drained out. These experiments are conducted with the PVC pipe sitting on a level wooden stand that allows water to pass through into a bucket; however, the PVC could be held on a variety of common lab items (a ring stand, cinderblocks, etc.). It is only important that the pipe is level and steady. To level our pipe, we used a standard bubble level. The apparatus we designed to hold the PVC pipe in place is a wooden box that fits the bucket, as shown in Fig. 4.

In this experiment, the motion of the falling water level is recorded using an ultrasonic motion detector.[12] The motion detector is placed in a three-fingered clamp attached to a ring stand approximately 40 cm above the top of the PVC pipe. It is important that the motion detector have a clear view of the water level. The motion detector is set up to collect a data point every half second. We discovered that the PVC cylinder needed to be 6 in in diameter so that the motion detector could "see" the water level and not the top edge of the cylinder.

This paper explores drainage through holes of four different diameters with two edges: sharp edge (no grommet) and rounded edges (grommet). The rounded edge case consists of a brass grommet[13] attached with a waterproof adhesive to a test cap with a hole that is slightly larger than the inner diameter of the grommet itself. Grommets have an extended length under the flange and a curved transition from flange to extension, as shown in Fig. 5. The length under the flange and the inner diameter are provided from the grommet manufacturer and checked with a measurement using calipers. The radius of curvature is measured approximately through the use of a Starrett radius gauge set. Dimensions of both the grommet and no grommet holes and their corresponding cap numbers are presented in Table I.

In general, the holes in the caps are plugged with tape or a rubber stopper to keep the fluid from draining before the desired time. We wait 10 minutes after adding fluid to the pipe to allow the fluid to settle, ensuring a smooth drain. Excess turbulence in the fluid during draining results in the formation of a vortex that seriously skews data collection and is an effect that is outside the scope of this investigation. When the fluid is still, data collection is begun on the Logger Pro software and the plug is removed from the cap. It is important that the pipe is disturbed as little as possible during this process. Bumping or jostling the fluid will cause ripples or sloshing, resulting in an inaccurate position measurement.

## Results & discussion of data

In our experiments, the velocity data tended to be noisy. This noise is attributed to the slow velocity of the top surface of the water. The motion detector confuses the slow motion of the top surface for no motion, and thus the velocity suddenly becomes zero at points, resulting in jagged data. Thus we choose to analyze the position data rather than the velocity data, fitting Eq. (8). The initial height, final height, and

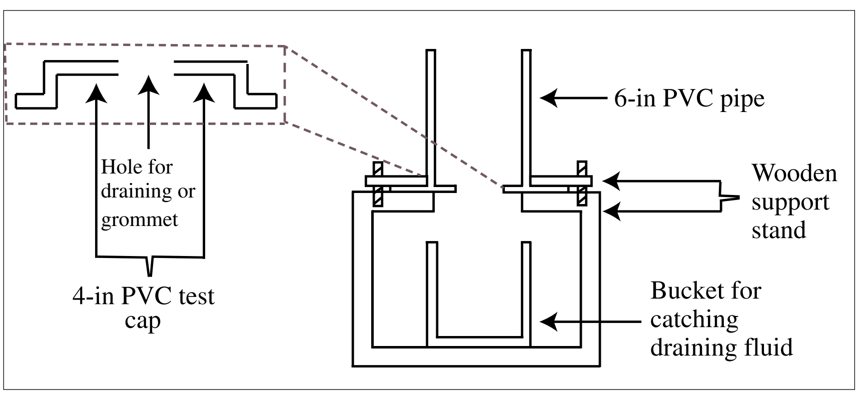


**Fig. 4. Schematic of apparatus designed to measure the draining of a fluid.**

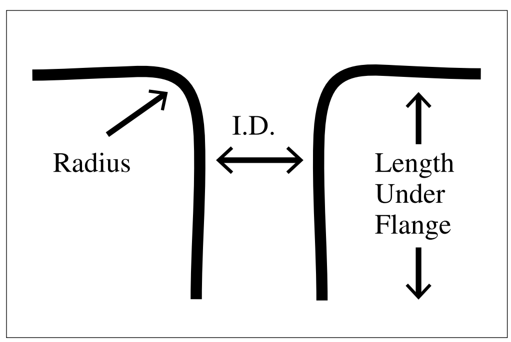


**Fig. 5. Grommet design showing its inner diameter (I.D.), length under flange, and radius of curvature.**

**Table I. Properties of drainage holes.**

| Cap Number | No Grommet | Grommet | | |
|---|---|---|---|---|
| | Inner Diameter (mm) | Inner Diameter (mm) | Length Under Flange (mm) | Radius of Curvature (mm) |
| 1 | 4.29 | 4.43 | 3.43 | 1.98 |
| 2 | 5.89 | 6.04 | 4.45 | 1.58 |
| 3 | 8.92 | 9.11 | 5.46 | 2.78 |
| 4 | 12.5 | 12.6 | 9.5 | 3.57 |

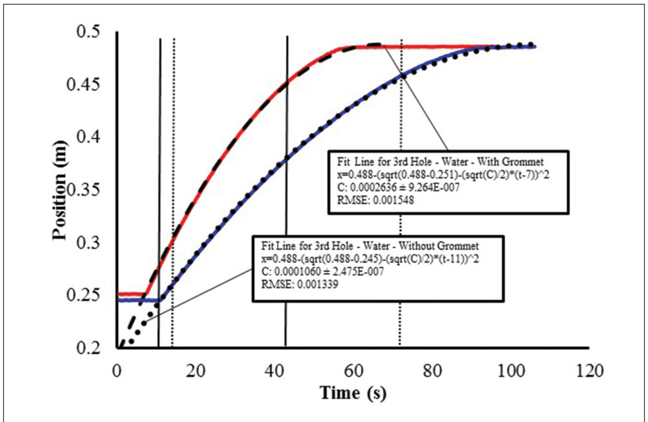


**Fig. 6. Sample of data taken with fit lines overlaid.**

start time are put in as constants, and Logger Pro determines the most appropriate value for the constant $C$.

Our physical model suggests there are no specific criteria for choosing which data to be fitted. However, in general, we chose data ranging from 5 s after the flow began to 10 s before the flow transitioned to a dripping state, which occurs 5 to 10 s before the data plateau out. The reason for this choice is to remove from consideration flow that is subject to start-up or dripping-state physics,[14-17] which is not contained in the physical model presented in the "Mathematical Models" section. A sample of the data collected for the grommet and no grommet case for hole #3 and the fit lines for both situations is shown in Fig. 6. The fit range, described above, is marked on the graph by a set of solid black lines (grommet case) and a set of dashed lines (no grommet case).

The data in Fig. 6 are the position of the water level versus time. The slope of the plotted line at any point in time represents the velocity of the water in the cylinder drained at that time. For both cases, the slope of the line at the beginning of the drain is large, indicating a high draining velocity. Likewise for both cases, the slope towards the end of the draining time is shallower. This is expected because as the depth of the water decreases, the pressure that forces the water out of the bottom orifice decreases as well. Less pressure forcing the water downward out of the orifice results in a slowing of the velocity. It is clear from Fig. 6 that the orifice with a rounded edge drains faster than the sharp-edged orifice, even though the holes are approximately the same diameter.

To quantify the effects of the vena contracta on fluid draining from an open-topped container, engineers use the coefficient of discharge. The coefficient of discharge is defined as a ratio of actual and theoretical mass flow rates.[18-19]

$$C_d = \frac{\textit{actual mass flow rate}}{\textit{theoretical mass flow rate}}. \tag{9}$$

The mass flow rate can be represented as the density of the fluid $\rho$ multiplied by the fluid velocity $v$ and the cross-sectional area of the flow $A$. It is important to note that here we make a distinction between the experimental area of the flow, $A_{2\exp}$, and the actual area of the orifice, $A_2$. Thus, in our case, the coefficient of discharge is:

$$C_d = \frac{A_{2\exp} v_{2\exp}}{A_2 v_2}, \tag{10}$$

where the fluid density $\rho$ has canceled out. By manipulation of Bernoulli's equation and the equation of continuity, we can develop an expression for the velocities, which can be substituted into the coefficient of discharge definition and simplified to produce

$$C_d = \frac{A_{2\exp}}{A_2}\sqrt{\frac{1-A_2^2/A_1^2}{1-A_{2\exp}^2/A_1^2}}. \tag{11}$$

Equation (11) shows that the coefficient of discharge is a product of two terms—the first is a ratio of the experimental to theoretical areas, and the second is a term that arises from a ratio of velocities. The first term is called the coefficient of contraction $C_c$ and the second is called the coefficient of velocity $C_v$. Hence, we may write $C_d = C_c C_v$. Equation (11) allows us to calculate the values for $C_c$ and $C_v$ separately and use those values to determine the coefficient of discharge for the system. Recall that from the fit constant $C$, we can determine the experimental area of the flow to be

$$A_{2\exp} = \frac{A_1}{\sqrt{\frac{2g}{C}+1}}. \tag{12}$$

We calculate the experimental values for $C_c$, $C_v$, and $C_d$ from the average of five runs for each hole. We find that $C_v$ ranges between $1.0\pm10^{-9}$ to $1.0\pm10^{-4}$, which is essentially a value of 1. A value of 1 makes sense in our setup because the area of the orifice is much smaller than the area of the top of the pipe. The value of 1 for the coefficient of velocity is also confirmed by literature references.[19] This being the case, the coefficient of discharge is effectively equal to the coefficient of contraction for the system. For simplicity, we represent only the final value for the coefficient of discharge, taken as an average from five runs each for both the grommet and no grommet cases. Again, the range for the fit lines from which these values were calculated is approximately 5 s after the drain starts to 5 to 10 s before the data plateau, as illustrated in Fig. 6. Values for the coefficient of discharge, including standard deviations, are shown in Table II.

We can now compare these experimental values for the coefficient of discharge to the expected values given in several textbooks and articles. The no grommet case, with a value for the coefficient of discharge of ~0.65, represents a sharp-edged orifice. The value of $C_d$ for a sharp-edged orifice was reported by Vennard and Street[19] as 0.61. Judd and King[1] measured values for $C_d$ between 0.59 and 0.61 depending on the size of the orifice and the depth of the water in the system. A rounded orifice, represented by the grommet case with an average coefficient of discharge of ~0.98, has reported values[19] of 0.98 or simply 1.00. In systems with a value for $C_d$ approaching 1.00, like the rounded grommet case, the actual area of the draining stream is closer to the measured area of the orifice. These systems are more accurately represented by the traditional draining-bucket theoretical calculations as presented in the "Mathematical Model" section of this paper. The vena contracta of a rounded orifice, then, can be assumed to be smaller than the vena contracta of a sharp-edged orifice, as stated above.

**Table II. Average experimental values after five runs, over a range of exit velocities. The coefficient is a dimensionless variable.**

| Cap Number | Coefficient of Discharge | |
|---|---|---|
| | With Grommet | Without Grommet |
| 1 | 1.03 ± 0.01 | 0.761 ± 0.003 |
| 2 | 0.984 ± 0.001 | 0.654 ± 0.002 |
| 3 | 0.970 ± 0.004 | 0.650 ± 0.003 |
| 4 | 0.96 ± 0.01 | 0.666 ± 0.005 |

The discharge coefficient's dependence on the flow's exit velocity is not so apparent in this setup or in our data. Our decision to fit the majority of the position-versus-time graph for a run results in a single composite value for the discharge coefficient during that run. If we had better data for the velocity of the water level in the container, we could experimentally calculate the discharge coefficient directly.

## Conclusion

With this project we have shown that there is an accurate way to measure and determine the coefficient of discharge for fluid flow that follows Bernoulli's law in an open container with orifices of different diameters. Measurements and calculations for the coefficient of discharge can be done with water only, or the fluid properties can be varied by changing the surface tension and/or viscosity of the fluids. We didn't report our investigations using cooking oil and acetone in this setup. Further research in this subject could investigate an improved method for physically measuring the area of the flow out of the cylinder (photography, video, force probe, etc.). Still the experiment presented here is accurate and informative, and it can be executed in any calculus-based college physics class with minimal expense and difficulty. Future work could investigate fluids draining from shaped containers.

## References


1. H. Judd and R. S. King, "Some experiments on the frictionless orifice," *Eng. News.* **56** (13), 326–331 (Sept. 1906).
2. G. L. Tuve and R. E. Sprenkle, "Orifice discharge coefficients for viscous liquids," *Instruments* **6**, 201–206 (Nov. 1933).
3. F. W. Medaugh and G. D. Johnson, "Investigation of the discharge and coefficients of small circular orifices," *Civil Eng.* **10** (7), 422–424 (July 1940).
4. J. H. Lienhard (IV) and J. H. Lienhard (V), "Velocity coefficients for free jets from sharp-edged orifices," *J. Fluids Eng.* **106**, 13–17 (March 1984).
5. J. N. Libii, "Mechanics of the slow draining of a large tank under gravity," *Am. J. Phys.* **71**, 1204–1207 (Nov. 2003).
6. M. E. Saleta, D. Tobia, and G. Salvador, "Experimental study of Bernoulli's equation with losses," *Am. J. Phys.* **73** (7), 598–602 (July 2005).
7. P. L. L. Escamilla, "Surface tension influence in vessel discharge: Comment on 'Experimental study of Bernoulli's equation with losses,' by Martin Eduardo Saleta, Dina Tobia, and Salvador Gil [*Am. J. Phys.* **73** (7), 598–602 (2005)]," *Am. J. Phys.* **77** (5), 477–478 (May 2009).
8. D. Guerra et al., "A Bernoulli's law lab in a bottle," *Phys. Teach.* **34**, 456–459 (Oct. 2005).
9. D. Guerra et al., "An introduction to dimensionless parameters in the study of viscous fluid flows," *Phys. Teach.* **49**, 175–179 (March 2011).
10. H. Lamb, *Hydrodynamics*, 6th ed. (Dover Publications, New York, 1945)), pp. 23–25.
11. L. M. Milne-Thompson, *Theoretical Hydrodynamics*, 4th ed. (Macmillan, New York, 1960), pp. 20–21.
12. Typical ultrasonic motion detectors can be purchased from laboratory supply companies like Vernier (http://www.vernier.com/) or PASCO (http://www.pasco.com/). We used the Vernier Logger Pro Motion Detector (Model MD-BTD). This version of the Logger Pro motion detector is ideal because the resolution is such that the detector can resolve the motion of the water without excess reflections from the edge of the PVC pipe.
13. The grommets used in this experiment are brass and, with the exception of the smallest grommet size, can be purchased at a local hardware store (General Tools, part numbers 1261-4, 1261-2, and 71260). The smallest grommet size is purchased from McMaster-Carr (http://www.mcmaster.com/), product number 9604K21.
14. J. Eggers, "Nonlinear dynamics and breakup of free-surface flows," *Rev. Mod. Phys.* **69** (3), 865–930 (July 1997).
15. Christophe Clanet and Juan C. Lasheras, "Transition from dripping to jetting," *J. Fluid Mech.* **383**, 307–326 (1999).
16. H. Henein, "The transition from free stream flow to dripping in draining vessels," *Can. Metall. Q.* **44** (2), 261–264 (2005).
17. V. Grubelnik and M. Marhl, "Drop formation in a falling stream of liquid," *Am. J. Phys.* **73** (5), 415–419 (May 2005).
18. Robert W. Fox and Alan T. McDonald, *Introduction to Fluid Mechanics*, 4th ed. (Wiley, New York, 1992), p. 379.
19. John K. Vennard and Robert L. Street, *Elementary Fluid Mechanics*, 5th ed. (Wiley, New York, 1975), p. 558.



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