Generation in the Bingo Closure

J. Beyerl*, Robert E. Jamison, J. Bowman Light, Clemson University
Bingo Closure, 5x5 case

- 5x5 playing board
- A square $s$ is dependent on a set $S$ when $s$ completes a line.
Bingo Closure, 5x5 case

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Bingo Closure, 5x5 case

- 5x5 playing board
- A square \( s \) is dependent on a set \( S \) when \( s \) completes a line.
Bingo Closure, 5x5 case

• $S$ is closed when no squares that are dependent on $S$ are not in $S$. 

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Bingo Closure, 5x5 case

- S is closed when no squares that are dependent on S are not in S.
Bingo Closure, 5x5 case

- The closure of $S$ is $S$ union all dependent squares, iterated until $S$ is closed.
Bingo Closure, 5x5 case

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Bingo Closure, 5x5 case

- The closure of $S$ is $S$ union all dependent squares, iterated until $S$ is closed.
Bingo Closure, 5x5 case

• $S$ is independent when no square in $S$ is dependent on $S$.

• *Don’t need this slide?*
Bingo Closure, 5x5 case

- A generating set for the board is a set $S$ whose closure is the entire board.

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  ● ● ●  
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  ● ● ● ●  
  ● ● ● ●  
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```
Bingo Closure, 5x5 case

• A generating set for the board is a set $S$ whose closure is the entire board.
Bingo Closure, 5x5 case

• The depth of $S$ the number of iterations required to close $S$. 

```
  ● ● ● ● ●
         ● ● ● ● ●
         ● ● ● ● ●
         ● ● ● ● ●
  ● ● ● ● ●
```
Bingo Closure, 5x5 case

• The depth of $S$ the number of iterations required to close $S$. 

```
  ● ● ● ● ● 1
  ● ● ● ● ●  
  ● ● ● ● ●  
  ● ● ● ● ●  
  ● ● ● ● ● 1
```
Bingo Closure, 5x5 case

- The depth of $S$ the number of iterations required to close $S$. 

```
  ● ● ● ● ● 1
  ● ● ● ● 2
  ● ● ● ● 1
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Bingo Closure, 5x5 case

- The depth of $S$ the number of iterations required to close $S$.

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Bingo Closure, 5x5 case

• What is the maximum depth of a set $S$?
• Easy upper bound: 12
Bingo Closure, 5x5 case

• What is the maximum depth of a set $S$?
• Easyish lower bound: 8
Bingo Closure, 5x5 case

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Bingo Closure, 5x5 case

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Bingo Closure, 5x5 case

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Bingo Closure, 5x5 case

• What is the maximum depth of a set $S$?

• Easyish lower bound: 8
Bingo Closure, 5x5 case

• What is the maximum depth of a set $S$?
• Easyish lower bound: 8
Bingo Closure, 5x5 case

- What is the maximum depth of a set \( S \)?
- Easyish lower bound: 8
Bingo Closure, 5x5 case

• What is the maximum depth of a set $S$?
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Bingo Closure, 5x5 case

- What is the maximum depth of a set $S$?
- Easyish lower bound: 8

```
  3  •  •  •  •  2
  •  8  8  6  •
  •  8  7  •  •
 4  •  •  5  •
  •  •  •  •  1
```
Bingo Closure, 5x5 case

• What is the maximum depth of a set $S$?
• Actually: 10
Bingo Closure, 5x5 case

• What is the maximum depth of a set \( S \)?
• Actually: 10
Bingo Closure, 5x5 case

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Bingo Closure, 5x5 case

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```
  ●  ●  3  ●  2
  ●  ●  ●   ●
  ●  ●  ●   ●
  ●  5  4  ●  ●
  7  6  ●  ●  1
```
Bingo Closure, 5x5 case

• What is the maximum depth of a set \( S \)?
• Actually: 10
Bingo Closure, 5x5 case

- What is the maximum depth of a set $S$?
- Actually: 10

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**Bingo Closure, 5x5 case**

- What is the maximum depth of a set $S$?
- Actually: 10

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Bingo Closure, 5x5 case

- What is the maximum depth of a set $S$?
- Actually: 10
- ...And this is optimal.

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Bingo Closure, 7x7 case

• A depth of 14 is optimal
• The previous solution to the 5x5 does not easily generalize because it started on a diagonal.
Bingo Closure, 7x7 case

- A depth of 14 is optimal
- Use a 5x5 solution that starts off the diagonal
- ...which requires only a slight readjustment
Bingo Closure, 7x7 case

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```

```
Bingo Closure, 7x7 case

- A depth of 14 is optimal
- Use a 5x5 solution that starts off the diagonal
- ...which requires only a slight readjustment
- ...and now it starts on a vertical
Bingo Closure, 7x7 case

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Bingo Closure, 7x7 case

• A depth of 14 is optimal
• Use a 5x5 solution that starts off the diagonal
• ...which requires only a slight readjustment
• ...and now it starts on a vertical
Bingo Closure, 7x7 case

- A depth of 14 is optimal
- Use a 5x5 solution that starts off the diagonal
- ...which requires only a slight readjustment
- ...and now it starts on a vertical

```
2  •  1  •  •
9  •  •  8  •
    •  •  9  •
  •  5  •  •  4
7  6  •  •  3
```
Bingo Closure, 7x7 case

- A depth of 14 is optimal
- Use a 5x5 solution that starts off the diagonal
- ...which requires only a slight readjustment
- ...and now it starts on a vertical

```
2  ●  1  ●  ●  ●
9  ●  ●  8  ●  ●
10 ●  ●  9  ●  ●
●  5  ●  ●  4  ●
7  6  ●  ●  3  ●
```
Bingo Closure, 7x7 case

- A depth of 14 is optimal
- Fill in the rest avoiding the starting vertical
Bingo Closure, 7x7 case

- A depth of 14 is optimal
- Fill in the rest avoiding the starting vertical
- ...and we can get the corners for free
Bingo Closure, 7x7 case

- A depth of 14 is optimal
- Fill in the rest avoiding the starting diagonal
- ...and we can get the corners for free

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Bingo Closure, nxn case, n odd

- A depth of 2n is optimal
- Continue this pattern

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Bingo Closure, nxn case, n odd

• A depth of 2n is optimal
• Continue this pattern
Bingo Closure, nxn case, n even

- A depth of 2n is optimal
- A similar pattern, based off a solution to the 6x6 case
Bingo Closure, nxn case, n even

- A depth of $2n$ is optimal
- A similar pattern, based off a solution to the 6x6 case
Bingo Closure, 2x2 case

• A depth of 1 is trivially optimal.
Bingo Closure, 3x3 case

• A depth of 4 is optimal
• Construction gives lower bound
• Upper bound by exhaustion is easy.
Bingo Closure, 4x4 case

• A depth of 5 is optimal
• Construction gives lower bound
• Upper bound requires more attention...
• Only case in which a set of maximum depth is not a generating set

```
  5   1   4   1
  4   ●   3   ●
  3   ●   ●   2
  ●   ●   ●   2
  5   4   ●   ●
  3   ●   ●   ●
  ●   ●   ●   1
```
Future Work

• Investigate other properties
• Different board sizes