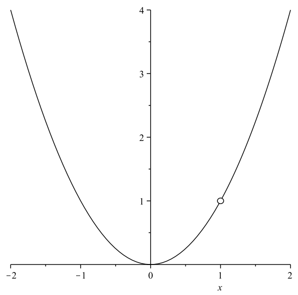
Math 1591 - Sample Test 1 Solutions

1. Calculate $\lim_{x\to 1} \frac{x^3 - x^2}{x-1}$ using the techniques of graphically, numerically and analytically.

(i) Graphically



from which the graph says $\lim_{x\to 1} \frac{x^3 - x^2}{x-1} = 1$.

(ii) Numerically

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.8100	0.9801	0.9980	1.0020	1.0201	1.2100

from which the table says $\lim_{x\to 1} \frac{x^3 - x^2}{x-1} = 1$.

(iii) Analytically

$$\lim_{x\to 1} \frac{x^3 - x^2}{x-1} = \lim_{x\to 1} \frac{x^2\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x\to 1} x^2 = 1$$

2. Calculate the following limits analytically.

(i)

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4} \frac{\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{x-4}} = \lim_{x\to 4} \sqrt{x}+2 = 4$$

(ii)

$$\lim_{x\to 0} \frac{\tan 4x}{\tan 2x} = \lim_{x\to 0} \frac{\sin 4x}{\cos 4x} \cdot \frac{\cos 2x}{\sin 2x} = \lim_{x\to 0} \frac{\sin 4x}{x} \cdot \frac{x}{\sin 2x} \cdot \frac{\cos 2x}{\cos 4x} = 4 \cdot \frac{1}{2} \cdot 1 = 2$$

(iii)

$$\lim_{x\to\infty}\frac{3x^2+4}{x^2+2x+1} = \lim_{x\to\infty}\frac{3+\frac{4}{x^2}}{1+\frac{2}{x}+\frac{1}{x^2}} = \frac{3+0}{1+0+0} = 3$$

3. Calculate the first derivative (either $f'(x)$ or y') of the following. Do not simplify your answer

(i)

$$\begin{aligned} f(x) &= x\sqrt{x}+\frac{1}{x\sqrt{x}} \qquad f'(x) = \frac{3}{2}x^{1/2}-\frac{3}{2}x^{-5/2} \\ &= x^{3/2}+x^{-3/2} \end{aligned}$$

(ii)

$$y = \frac{4e^x}{x^2+1}, \quad y' = \frac{4e^x(x^2+1)-4e^x\cdot 2x}{(x^2+1)^2}$$

(iii)

$$y = x^2\tan x, \quad y' = 2x\tan x + x^2\sec^2 x$$

(iv)

$$\begin{aligned} f(x) &= \sin\left(\sqrt{4x^2+x}\right), \\ f'(x) &= \cos\left(\sqrt{4x^2+x}\right)\cdot\frac{1}{2}\left(4x^2+x\right)^{-1/2}\cdot(8x+1) \end{aligned}$$

(v)

$$\begin{gathered} x^2-x^3y^4+5y = x+y^2 \\ 2x-\left(3x^2y^4+4x^3y^3y'\right)+5y' = 1+2yy' \\ -4x^3y^3y'+5y'-2yy' = 1-2x+3x^2y^4 \\ y' = \frac{1-2x+3x^2y^4}{5-2y-4x^3y^3} \end{gathered}$$

3. Calculate the derivative of $f(x) = \dfrac{1}{x}$ from the definition.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h} \\ &= \lim_{h\to 0}\frac{\dfrac{x-(x+h)}{x(x+h)}}{h} \\ &= \lim_{h\to 0}\frac{-\cancel{h}}{\cancel{h}x(x+h)} = -\frac{1}{x^2} \end{aligned}$$

4. Calculate the equation of the tangent to $y = x^4 - 2x^3 + 2x^2$ at $(1, 1)$.

So $y' = 4x^3 - 6x^2 + 4x$ and at $x = 1$ then $y' = 4 - 6 + 4 = 2$ so the equation of the tangent is

$$y - 1 = 2(x - 1)$$

5. Consider the function

$$f(x) = \begin{cases} x^2, & x \le 0 \\ x^3, & x > 0 \end{cases}$$

Is $f(x)$ continuous at $x = 0$? Is $f(x)$ differentiable at $x = 0$? What is $f'(0)$ if it exists.

(i) Continuity

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} x^2 = 0 \\ \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} x^3 = 0 \\ f(0) &= 0^2 = 0 \end{aligned}$$

Since $\lim_{x \to 0} f(x) = f(0)$, then yes, $f(x)$ is continuous at $x = 0$.

(i) Differentiability

$$\begin{aligned} \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0^-} \frac{x^2}{x} = 0 \\ \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0^+} \frac{x^3}{x} = 0 \end{aligned}$$

Since these are equation, then yes, $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$.

7. Using the $\delta - \varepsilon$ definition, prove

$$\lim_{x \to 2} 2x - 1 = 3.$$

For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|(2x - 1) - 3| < \varepsilon \text{ whenever } 0 < |x - 2| < \delta$$

Aside		Proof	
$\lvert(2x-1)-3\rvert$	$< \varepsilon$	$\lvert x-2\rvert$	$< \varepsilon/2$
$\lvert 2x-4\rvert$	$< \varepsilon$	$2\lvert x-2\rvert$	$< \varepsilon$
$2\lvert x-2\rvert$	$< \varepsilon$	$\lvert 2x-4\rvert$	$< \varepsilon$
$\lvert x-2\rvert$	$< \varepsilon/2$	$\lvert(2x-1)-3\rvert$	$< \varepsilon$
choose δ	$= \varepsilon/2$	Thus, $\lvert f(x) - L\rvert$	$< \varepsilon$