Math 1591 - Sample Test 2 Solutions

1. The mean value theorem.

If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), there exists a c in (a,b) such that

$$\frac{f(b)-f(a)}{a-b} = f'(c)$$

(i) $f(x) = x^3 - x$ on $[0,2]$ so $\dfrac{f(2)-f(0)}{2-0} = \frac{6}{2} = 3$ $f'(x) = 3x^2 - 1$ so
$3c^2 - 1 = 3$ giving that $c = \frac{2}{\sqrt{3}}$ noting that $c = -\frac{2}{\sqrt{3}}$ is not in the interval.

(ii) $f(x) = \dfrac{x}{x+2}$ on $[1,10]$ so $\dfrac{f(10)-f(1)}{10-1} = \dfrac{1}{18}$, $f'(x) = \dfrac{2}{(x+2)^2}$ so
$\dfrac{2}{(c+2)^2} = \dfrac{1}{18}$ giving that $c = 4$ noting that $c = -8$ is not in the interval.

2. Using L'Hopital's rule calculate the follow.

(i) $\displaystyle \lim_{x\to 0} x\ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} x = 0$

(ii) $\displaystyle \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

(iii) Let $y = x^{\sin x}$ so $\ln y = \sin x \ln x = \dfrac{\ln x}{\csc x}$

so $\displaystyle \lim_{x\to 0} \frac{\ln x}{\csc x} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\csc x \cot x} = -\lim_{x\to 0} \frac{\sin^2 x}{x\cos x} = -\lim_{x\to 0} \frac{2\sin x\cos x}{\cos x - x\sin x} = 0$

Now since $x \to 0$ gives $\ln y \to 0$ gives $y \to 1$, thus, $\displaystyle \lim_{x\to 0} x^{\sin x} = 1$

3. Find the absolute min and max on the given interval

(i) $f(x) = 1 - x^2$ so $f'(x) = -2x$. Thus, $f'(x) = 0$ when $x = 0$ so

$$f(-1) = 0, f(0) = 1 \text{ (max)}, f(3) = -8 \text{ (min)}$$

(ii) $f(x) = 2x^3 - 15x^2 + 24x$ so $f'(x) = 6x^2 - 30x + 24 = 6(x-1)(x-4)$.
Thus, $f'(x) = 0$ when $x = 1$, 4 so

$$f(0) = 0, f(1) = 11 \text{ (max)}, f(3) = -9 \text{ (min)}$$

4. Find the roots of the following using Newton's method.

(i) $$f(x) = x^4 - 3x^2 - 2x - 1, \quad f'(x) = 4x^3 - 6x - 2$$

$$x_{n+1} = x_n - \frac{x_n^4 - 3x_n^2 - 2x_n - 1}{4x_n^3 - 6x_n - 2}$$

The graph suggest two roots. One around $x = -1.5$ and the other around $x = 2$

n	x_n
0	1.0000
1	-0.2500
2	-1.4652
3	-1.4481
4	-1.4476
5	-1.4476

n	x_n
0	2.0000
1	2.0556
2	2.0523
3	2.0523

(ii)
$$f(x) = \tan x - \sqrt{x}, \quad f'(x) = \sec^2 x - \frac{1}{2\sqrt{x}}$$

$$x_{n+1} = x_n - \frac{\tan x_n - \sqrt{x_n}}{\sec^2 x_n - \frac{1}{2\sqrt{x_n}}}$$

The graph suggests two roots. One is $x = 0$ and the other around $x = 0.7$

n	x_n
0	1.0000
1	0.8095
2	0.7126
3	0.6954
4	0.6949
5	0.6949

5. For the following, calculate
   (i) critical numbers
   (ii) increasing and decreasing
   (iii) min/max
   (iv) concavity
   (v) asymptotes
   (vi) then sketch and label

(a)
$$f(x) = x^4 - 10x^3 + 24x^2 + 32x - 128, \tag{1}$$

$$f'(x) = 4x^3 - 30x^2 + 48x + 32 = 2(2x+1)(x-4)^2, \tag{2}$$

$f'(x) = 0$ when $x = -1/2, 4$, critical points

$$f''(x) = 12x^2 - 60x + 48 = 12(x-1)(x-4) \tag{3}$$

$f''(x) = 0$ when $x = 1, 4$, possible points of inflection.

x		$-\frac{1}{2}$		1		4	
$2x+1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$(x-4)^2$	$+$	$+$	$+$	$+$	$+$	0	$+$
$(2x+1)(x-4)^2$	$-$	0	$+$	$+$	$+$	0	$+$
$slope$	$\backslash$	$-$	$/$	$/$	$/$	$-$	$/$
$x-1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	$-$	$-$	0	$+$
$(x-1)(x-4)$	$+$	$+$	$+$	0	$-$	0	$+$
h/v	$\smile$	$\smile$	$\smile$	pi	$\frown$	pi	$\smile$

(i) crit. # $x = -\frac{1}{2},\ x = 4$

(ii) inc. $(-\frac{1}{2}, 4)(4, \infty)$

(ii) dec. $(-\infty, -\frac{1}{2})$

(iii) min $(-\frac{1}{2}, -\frac{2187}{16})$

(iii) max - none

(iv) pt. inflect. $(1, -81)$

(iv) concave up $(-\infty, 1), (4, \infty)$

(iv) concave down $(1, 4)$

(v) asymptotes - none

(b)

$$f(x) = x^3 - 3x^2 - 9x, \tag{4}$$

$$f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1), \tag{5}$$

$f'(x) = 0$ when $x = -1, 3$, critical points

$$f''(x) = 6x - 6 = 6(x-1) \tag{6}$$

$f''(x) = 0$ when $x = 1$, possible points of inflection.

x		-1		1		3	
$x-3$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x+1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$(x-2)(x+1)$	$+$	0	$-$	$-$	$-$	0	$+$
$slope$	$/$	$-$	$\backslash$	$\backslash$	$\backslash$	$-$	$/$
$x-1$	$-$	$-$	$-$	0	$+$	$+$	$+$
h/v	$\frown$	$\frown$	$\frown$	pi	$\smile$	$\smile$	$\smile$

(i) crit. # $x = -1,\ x = 3$

(ii) inc. $(-\infty, -1), (3, \infty)$

(ii) dec. $(-1, 3)$

(iii) max $(-1, 5)$

(iii) min $(3, -27)$

(iv) pt. inflect. $(1, -11)$

(iv) concave up $(1, \infty)$

(iv) concave down $(-\infty, 1)$

6. A ladder 13 feet long is resting against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 ft/sec. At rate is the tip of the ladder moving down the wall when the base of the ladder is 5 ft away from the wall?

Soln: We are given $\frac{dx}{dt} = 2$ and we wish to find $\frac{dy}{dt}$ when $x = 5$. Relating variables gives $x^2 + y^2 = 13^2$ so relating rates gives $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ so $\frac{dy}{dt} = -\frac{x\frac{dx}{dt}}{y}$. From $x^2 + y^2 = 13^2$, if $x = 5$, then $y = 12$ giving $\frac{dy}{dt} = -\frac{x\frac{dx}{dt}}{y} = -\frac{5}{6}$ ft/sec.

7. A paper cup in the shape of an inverted cone with height 10 cm and a base of radius 3 cm, is being filled at a rate of 2 cm^3/min. Find the rate of change in the height of the water when the height of the water is 5 cm.

Soln: We are given $\frac{dV}{dt} = 2$ and we wish to find $\frac{dh}{dt}$ when $h = 5$. Relating variables gives $V = \frac{1}{3}\pi r^2 h$ and by similar triangles $\frac{r}{3} = \frac{h}{10}$ so $V = \frac{1}{3}\pi\left(\frac{3h}{10}\right)^2 h = \frac{3\pi h^3}{100}$. Relating rates gives $\frac{dV}{dt} = \frac{9\pi h^2}{100}\frac{dh}{dt}$ or $\frac{dh}{dt} = \frac{100}{9\pi h^2}\frac{dV}{dt}$ and when $h = 5$ gives $\frac{dh}{dt} = \frac{8}{9\pi}$ cm/min.

8. A rancher has 200 feet of fencing with which to enclose two adjacent rectangular corrals. What dimensions should be used so that the enclosed area will be a maximum?

Soln: If we let x and y be the length and height of one corral, then the perimeter is $P = 4x + 3y = 200$ and the total area is $A = 2xy = \frac{200y - 3y^2}{2}$. Thus, $A' = \frac{200 - 6y}{2}$ and $A' = 0$ when $y = 100/3$. Further $A'' = -3 < 0$ when $y = 100/3$ giving a maximum. Thus, $x = 25$ and the dimensions of one of the two corrals is $100/3' \times 25'$.

9. An box with a square bottom is to be built that holds 64 cubic feet. Find the dimensions of the box that will minimize the surface area of the box.

Soln: If we let x be one side of the square base and y be the height of the box, then the volume is $V = x^2 y = 64$ and the total area is $A = 2x^2 + 4xy = 2x^2 + \frac{4 \cdot 64}{x}$. Thus, $A' = 4x - \frac{4 \cdot 64}{x^2}$ and $A' = 0$ when $x = 4$. Further $A'' = 4 + \frac{2 \cdot 4 \cdot 64}{x^3} > 0$ for $x = 4$ giving a minimum. Thus, $y = 4$ and the dimensions of the box is $4' \times 4' \times 4'$.

10. A company needs to lay cable from point A on one side of a river 2 miles wide to point B on the other side of the river, 5 miles downstream. If the cost to lay the cable under water is \$100 per mile and to lay the cable underground is \$60 per mile, at what point on the other side of the river should you lay the cable such that the cost will be a minimum?

Soln: If we let x be the spot where we first reach across the river, then by the Pythagorean thm. the distance underwater is $\sqrt{x^2+4}$ while the distance where we lay the cable under ground is $5-x$. Thus, the cost is $C = 100\sqrt{x^2+4} + 60(5-x)$ and $C' = \dfrac{100x}{\sqrt{x^2+4}} - 60$. Solving $C' = 0$ gives $x = 3/2$. Further $C'' = \dfrac{400}{(x^2+4)^{3/2}} > 0$ for all C giving a minimum. Therefore, the point is $3/2$ miles downstream.