Nonlinear PDE Example

Solve $xu_xu_y + yu_y^2 = 1, \quad u(x, -x^2) = 1$

If we set $F = xp\,q + y\,q^2 - 1$ then

$$F_x = pq, \quad F_y = q^2, \quad F_u = 0, \quad F_p = xq, \quad F_q = xp + 2yq.$$

The characteristic equations thus becomes

$$x_s = F_p = xq, \tag{1a}$$
$$y_s = F_q = xp + 2yq, \tag{1b}$$
$$u_s = pF_p + qF_q = 2(xpq + yq^2) = 2, \tag{1c}$$
$$p_s = -F_x - pF_u = -pq, \tag{1d}$$
$$q_s = -F_y - qF_u = -q^2. \tag{1e}$$

To these we must associate BC's. We will let $y = -x^2$ in the $x - y$ plane be $s = r$ in the $r - s$ plane and connect the two boundaries by $x = r$. Thus, the new BC's are

$$x = r, \quad y = -r^2, \quad u = 1, \quad \text{on} \quad s = r.$$

In order to complete the system, we will need BC's for p and q. From the BC $u(x, -x^2) = 1$, differentiating wrt x gives

$$u_x - 2xu_y = 0 \quad \text{or} \quad p - 2rq = 0. \tag{2}$$

From the original PDE

$$rpq - r^2q^2 = 1. \tag{3}$$

Solving (2) and (3) for p and q gives two cases: $p = \pm 2,\ q = \pm 1/r$. We will only look at the positive case.

Solving (1e) gives

$$\frac{1}{q} = s + a(r),$$

and with $q = \frac{1}{r}$ when $s = r$ gives $a(r) = 0$. Thus

$$\frac{1}{q} = s. \tag{4}$$

Using (4) in (1d) gives

$$p_s = -\frac{p}{s} \quad \Rightarrow \quad p = \frac{b(r)}{s},$$

and the BC here gives $b(r) = 2$ so

$$p = \frac{2r}{s}. \tag{5}$$

Next we solve (1c) subject to $u(r,r) = 1$. We obtain

$$u = 2s + c(r) \text{ and with the BC } u = 2s + 1 - 2r. \tag{6}$$

From (1a)

$$x_s = xq = \frac{x}{s} \quad \Rightarrow \quad x = d(r)s.$$

Since $x = r$ when $s = r$ gives $d(r) = 1$ and thus

$$x = r. \tag{7}$$

Finally, we consider (1b)

$$y_s = xp + 2yq = 2r + \frac{2y}{s}.$$

This is a linear ODE which readily integrates giving

$$\frac{y}{s^2} = -\frac{2r}{s} + e(r),$$

and with the BC $y = -r^2$ when $s = r$ gives $e(r) = 1$ and so

$$y = s^2 - 2rs. \tag{8}$$

Eliminating r and s from (6), (7) and (8) gives the solution of our PDE as

$$u = x + \frac{y}{x} + 1. \tag{9}$$