

Math 4315 - PDEs

Sample Test 1 - Solutions

1. Solve the following first order PDEs by introducing an alternate coordinate system (*i.e.* $(x, y) \rightarrow (r, s)$)

$$(i) \quad xu_x - yu_y = 2u,$$

$$(ii) \quad yu_x - u_y = 1,$$

Solutions

(i) If $u_s = u_x x_s + u_y y_s$ then choosing

$$x_s = x, \quad y_s = -y, \quad \text{gives} \quad u_s = 2u.$$

Solving gives

$$x = a(r)e^s, \quad y = b(r)e^{-s}, \quad u = c(r)e^{2s},$$

where $a(r)$, $b(r)$ and $c(r)$ are arbitrary functions of r . Eliminating s from the first two and first and third gives

$$xy = a(r)b(r) = A(r), \quad \frac{u}{x^2} = c(r)/a(r)^2 = B(r).$$

Noting again that $A(r)$ and $B(r)$ are arbitrary. Further, elimination of r gives

$$\frac{u}{x^2} = f(xy) \quad \text{or} \quad u = x^2 f(xy).$$

(ii) If $u_s = u_x x_s + u_y y_s$ then choosing

$$x_s = y, \quad y_s = -1, \quad \text{gives} \quad u_s = 1.$$

Solving the second and third gives

$$y = -s + b(r), \quad u = s + c(r),$$

giving the first as

$$x_s = y = -s + b(r).$$

Integrating gives

$$x = -\frac{(b(r) - s)^2}{2} + a(r),$$

where $a(r)$, $b(r)$ and $c(r)$ are arbitrary functions of r . Eliminating s from x and y gives

$$x + \frac{y^2}{2} = a(r), \quad \text{or} \quad 2x + y^2 = A(r).$$

Eliminating s from y and u gives

$$u + y = c(r) + b(r) = B(r).$$

Noting again that $A(r)$ and $B(r)$ are arbitrary. Further, elimination of r gives

$$u + y = f(2x + y^2) \quad \text{or} \quad u = -y + f(2x + y^2).$$

2. Solve the following using the method of characteristics

$$\begin{aligned} (i) \quad yu_x + xu_y &= xu, \quad u(x, 0) = e^{-x^2} \\ (ii) \quad xu_x + 2uu_y &= u, \quad u(x, 0) = x^2 \end{aligned}$$

Solutions

(i) The characteristic equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{du}{xu}.$$

Solving the first pair, *i.e.*

$$\frac{dx}{y} = \frac{dy}{x} \quad \text{gives} \quad c_1 = x^2 - y^2.$$

Solving the second and third, *i.e.*

$$\frac{dy}{x} = \frac{du}{xu} \quad \text{gives} \quad y = \ln |u| - \ln |c_2| \quad \text{or} \quad c_2 = u e^{-y}.$$

The solution is therefore given by

$$u e^{-y} = f(x^2 - y^2) \quad \text{or} \quad u = e^y f(x^2 - y^2).$$

Imposing the initial condition $u(x, 0) = e^{-x^2}$ gives

$$u(x, 0) = e^0 f(x^2 - 0^2) = e^{-x^2} \implies f(x) = e^{-x}.$$

This gives the solution as

$$u = e^y e^{-(x^2-y^2)} = e^{y^2+y-x^2}.$$

(ii) The characteristic equations are

$$\frac{dx}{x} = \frac{dy}{2u} = \frac{du}{u}.$$

Solving the first and third, *i.e.*

$$\frac{dx}{x} = \frac{du}{u} \quad \text{gives} \quad \ln|x| = \ln|u| - \ln|c_2| \quad \text{or} \quad \frac{u}{x} = c_2.$$

Solving the second and third, *i.e.*

$$\frac{dy}{2u} = \frac{du}{u} \quad \text{gives} \quad \frac{y}{2} = u - c_1 \quad \text{or} \quad c_1 = u - \frac{y}{2}.$$

The solution is therefore given by

$$\frac{u}{x} = f\left(u - \frac{y}{2}\right) \quad \text{or} \quad u = x f\left(u - \frac{y}{2}\right).$$

Imposing the initial condition $u(x, 0) = x^2$ gives

$$x^2 = x f(x^2 - 0) \implies f(x^2) = x \implies f(x) = \sqrt{x}.$$

This gives the solution as

$$u = x \sqrt{u - \frac{y}{2}}.$$

3. Solve the following nonlinear PDE

$$(i) \quad xu_x^2 + u_y = 1, \quad u(x, 1) = x + 1.$$

$$(ii) \quad u_x u_y - 2xu_x - 2yu_y = 0, \quad u(x, 0) = x^2$$

Solution

(i) If $F = xp^2 + q - 1$ then the characteristic equations are

$$x_s = F_p = 2xp \tag{1.4a}$$

$$y_s = F_q = 1 \tag{1.4b}$$

$$u_s = pF_p + qF_q = 2xp^2 + q \tag{1.4c}$$

$$p_s = -(F_x + pF_u) = -p^2 \tag{1.4d}$$

$$q_s = -(F_y + qF_u) = 0. \tag{1.4e}$$

To these we associate the following initial condition. When $s = 0$, then

$$y = 1, \quad x = r, \quad u = r + 1, \quad p = 1, \quad q = 1 - r, \quad (1.5)$$

where p is obtained from differentiating the initial condition and q from the original equation.

From (1.4d) and (1.4e) we find that

$$\frac{1}{p} = s + A(r), \quad q = B(r).$$

From the boundary conditions (1.5) we find that $A = 1$ and $B = 1 - r$ so

$$\frac{1}{p} = s + 1, \quad q = 1 - r.$$

From (1.4a) and (1.4a) we integrate giving

$$x = C(r)(s + 1)^2, \quad y = s + D(r).$$

The boundary conditions (1.5) gives $A = r$ and $D = 1$. Thus,

$$x = r(s + 1)^2, \quad y = s + 1. \quad (1.6)$$

From (1.4c) we have

$$\begin{aligned} u_s &= 2xp^2 + q = r + 1 \Rightarrow \\ u &= (r + 1)s + E(r) = (r + 1)s + r + 1. \end{aligned} \quad (1.7)$$

Eliminate r and s from (1.6) and (1.7) gives the solution

$$u = y + \frac{x}{y}.$$

(ii) If $F = pq - 2xp - 2yq$ then the characteristic equations are

$$x_s = F_p = q - 2x \quad (1.8a)$$

$$y_s = F_q = p - 2y \quad (1.8b)$$

$$u_s = pF_p + qF_q = 2pq - 2xp - 2yq = pq \quad (1.8c)$$

$$p_s = -(F_x + pF_u) = 2p \quad (1.8d)$$

$$q_s = -(F_y + qF_u) = 2q. \quad (1.8e)$$

To these we associate the following initial condition. When $s = 0$, then

$$y = 0, \quad x = r, \quad u = r^2, \quad p = 2r, \quad q = 2r, \quad (1.9)$$

where p is obtained from differentiating the initial condition and q from the original equation. As we need p and q to find x , y and u , we focus on these first. Solving (1.4d) and (1.4e) for p and q gives

$$p = a(r)e^{2s}, \quad q = b(r)e^{2s},$$

and imposing the initial condition gives $a(r) = 2r$ and $b(r) = 2r$. Thus,

$$p = 2re^{2s}, \quad q = 2re^{2s}.$$

From (1.4c) we see that

$$u_s = 4r^2e^{4s},$$

which integrates giving

$$u = r^2e^{4s} + c(r),$$

and the initial condition here gives $c(r) = 0$. Substituting (1.10) into (1.4a) and (1.4b) gives

$$x_s + 2x - 2re^{2s} = 0, \quad y_s + 2y - 2re^{2s} = 0.$$

Solving yields

$$x = \frac{r}{2} e^{2s} + d(r) e^{-2s}, \quad y = \frac{r}{2} e^{2s} + e(r) e^{-2s}.$$

Applying the remaining initial conditions gives

$$r = \frac{r}{2} e^0 + d(r) e^0, \quad 0 = \frac{r}{2} e^0 + e(r) e^0,$$

showing that $d(r) = \frac{r}{2}$ and $e(r) = -\frac{r}{2}$. Thus, we have the following parametric solutions

$$x = \frac{r}{2} \left(e^{2s} + e^{-2s} \right), \quad y = \frac{r}{2} \left(e^{2s} - e^{-2s} \right), \quad u = r^2 e^{4s}.$$

Eliminate r and s gives

$$u = (x + y)^2.$$