

Math 4315 - PDEs

Sample Test 2 - Solutions

1. Transform the following PDEs to canonical form. In the case that the type is hyperbolic, transform to both modified and regular form

$$(i) \quad 6u_{xx} - 5u_{xy} + u_{yy} = 0,$$

$$(ii) \quad 4u_{xx} + 12u_{xy} + 13u_{yy} = 0,$$

$$(iii) \quad u_{xx} + 4u_{xy} + 4u_{yy} - u_x = 0,$$

$$(iv) \quad x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} - xu_x - yu_y = 0,$$

$$(v) \quad 2y^2u_{xx} - 5xyu_{xy} + 2x^2u_{yy} = 0,$$

$$(vi) \quad x^2u_{xx} - 2xu_{xy} + 55u_{yy} + xu_x = 0,$$

1(i) The PDE to transform is

$$6u_{xx} - 5u_{xy} + u_{yy} = 0.$$

Here $a = 6$, $b = -5$ and $c = 1$ so $b^2 - 4ac = 1 > 0$ so the PDE is hyperbolic.

For the modified hyperbolic form, it is necessary to solve

$$6r_x^2 - 5r_xr_y + r_y^2 = 0, \quad 6s_x^2 - 5s_xs_y + s_y^2 = 0,$$

noting that they factor

$$(2r_x - r_y)(3r_x - r_y) = 0, \quad (2s_x - s_y)(3s_x - s_y) = 0.$$

Thus, solving

$$2r_x - r_y = 0, \quad 3s_x - s_y = 0,$$

gives r and s as

$$r = x + 2y, \quad s = x + 3y.$$

The derivatives transform as follows:

$$\begin{aligned} (i) \quad u_x &= u_r + u_s, & u_y &= 2u_r + 3u_s, \\ (ii) \quad u_{xx} &= u_{rr} + 2u_{rs} + u_{ss}, \\ (iii) \quad u_{xy} &= 2u_{rr} + 7u_{rs} + 3u_{ss}, \\ (iv) \quad u_{yy} &= 4u_{rr} + 12u_{rs} + 9u_{ss}, \end{aligned}$$

thus transforming the PDE to modified hyperbolic form

$$-11u_{rs} = 0, \quad \Rightarrow \quad u_{rs} = 0.$$

For regular hyperbolic form, it is necessary to solve

$$6r_x^2 - 5r_x r_y + r_y^2 = 0, \quad 6s_x^2 - 5s_x s_y + s_y^2 = 0,$$

using the quadratic formula. Dividing by r_y^2 gives

$$6 \left(\frac{r_x}{r_y} \right)^2 - 5 \frac{r_x}{r_y} + 1 = 0,$$

and solving gives

$$\frac{r_x}{r_y} = \frac{5 \pm 1}{12}.$$

Therefore, we solve the first order PDE

$$r_x = \frac{5 \pm 1}{12} r_y \quad \text{or} \quad 12r_x - (5 \pm 1) r_y = 0.$$

Using the method of characteristics gives

$$\frac{dx}{12} = -\frac{dy}{5 \pm 1}; \quad dr = 0,$$

which leads to

$$\begin{aligned}r, s &= (5 \pm 1)x + 12y, \\r, s &= 5x + 12y \pm x,\end{aligned}$$

where we choose

$$r = 5x + 12, \quad s = x.$$

The derivatives transform as follows:

$$\begin{aligned}(i) \quad u_x &= 5u_r + u_s, \quad u_y = 12u_s, \\(ii) \quad u_{xx} &= 25u_{rr} + 10u_{rs} + u_{ss}, \\(iii) \quad u_{xy} &= 60u_{rr} + 12u_{rs}, \\(iv) \quad u_{yy} &= 144u_{rr},\end{aligned}$$

thus transforming the PDE to regular hyperbolic form

$$-150u_{rr} + 150u_{ss} = 0, \quad \Rightarrow \quad u_{rr} - u_{ss} = 0.$$

1(ii). The PDE to transform is

$$4u_{xx} + 12u_{xy} + 13u_{yy} = 0.$$

Here $a = 4$, $b = 12$ and $c = 13$ so $b^2 - 4ac = -64 < 0$ so the PDE is elliptic.

Therefore, it is necessary to solve

$$4r_x^2 + 12r_xr_y + 13r_y^2 = 0, \quad 4s_x^2 + 12s_xs_y + 13s_y^2 = 0,$$

using the quadratic formula. Dividing by r_y^2 gives

$$4\left(\frac{r_x}{r_y}\right)^2 + 12\frac{r_x}{r_y} + 13 = 0,$$

and solving gives

$$\frac{r_x}{r_y} = \frac{-12 \pm 8i}{8} = \frac{-3 \pm 2i}{2}.$$

Therefore, we solve the first order PDE

$$r_x = \frac{-3 \pm 2i}{2} r_y \quad \text{or} \quad 2r_x - (-3 \pm 2i) r_y = 0.$$

Using the method of characteristics gives

$$\frac{dx}{2} = -\frac{dy}{-3 \pm 2i},$$

which leads to

$$\begin{aligned} r, s &= (-3 \pm 2i)x + 2y, \\ r, s &= 3x - 2y \pm 2x. \end{aligned}$$

where we choose

$$r = 3x - 2y, \quad s = 2x.$$

The derivatives transform as follows:

$$\begin{aligned} (i) \quad u_x &= 3r + 2u_s, \quad u_y = -2u_r, \\ (ii) \quad u_{xx} &= 9u_{rr} + 12u_{rs} + 4u_{ss}, \\ (iii) \quad u_{xy} &= -6u_{rr} - 4u_{rs}, \\ (iv) \quad u_{yy} &= 4u_{rr}, \end{aligned}$$

thus transforming the PDE to elliptic form

$$16u_{rr} + 16u_{ss} = 0, \quad \Rightarrow \quad u_{rr} + u_{ss} = 0.$$

1(iii). The PDE to transform is

$$u_{xx} + 4u_{xy} + 4u_{yy} - u_x = 0.$$

Here $a = 1$, $b = 4$ and $c = 4$ so $b^2 - 4ac = 0$ so the PDE is parabolic

Therefore, it is necessary to solve

$$r_x + 2r_y = 0,$$

Using the method of characteristics gives

$$\frac{dx}{1} = \frac{dy}{2},$$

which leads to

$$r = 2x - y, \quad s = y,$$

noting that the choice for s is arbitrary. The derivatives transform as follows:

$$(i) \quad u_x = 2u_r, \quad u_y = -u_r + u_s,$$

$$(ii) \quad u_{xx} = 4u_{rr},$$

$$(iii) \quad u_{xy} = -2u_{rr} + 2u_{rs},$$

$$(iv) \quad u_{yy} = u_{rr} - 2u_{rs} + u_{ss},$$

thus transforming the PDE to parabolic form

$$u_{ss} - \frac{u_r}{2} = 0.$$

1(iv). The PDE to transform is

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} - xu_x - yu_y = 0.$$

Here $a = x^2$, $b = 2xy$ and $c = y^2$ so $b^2 - 4ac = 0$ so the PDE is parabolic

Therefore, it is necessary to solve

$$x^2 r_x + xy r_y = 0, \quad \Rightarrow \quad xr_x + yr_y = 0.$$

Using the method of characteristics gives

$$\frac{dx}{x} = \frac{dy}{y}; dr = 0$$

which leads to

$$r = \frac{y}{x}, \quad s = y,$$

noting that the choice for s is arbitrary. The derivatives transform as follows:

$$\begin{aligned} (i) \quad u_x &= -\frac{y}{x^2}u_r, & u_y &= \frac{1}{x}u_r + u_s, \\ (ii) \quad u_{xx} &= \frac{y^2}{x^4}u_{rr} + \frac{2y}{x^3}u_r, \\ (iii) \quad u_{xy} &= -\frac{y}{x^3}u_{rr} - \frac{y}{x^2}u_{rs} - \frac{1}{x^2}u_r, \\ (iv) \quad u_{yy} &= \frac{1}{x^2}u_{rr} + \frac{2}{x}u_{rs} + u_{ss}, \end{aligned}$$

thus transforming the PDE to parabolic form

$$s^2u_{ss} - su_s = 0 \quad \Rightarrow \quad u_{ss} - \frac{1}{s}u_s = 0.$$

1(v) The PDE to transform is

$$2y^2u_{xx} - 5xyu_{xy} + 2x^2u_{yy} = 0.$$

Here $a = 2y^2$, $b = -5xy$ and $c = 2x^2$ so $b^2 - 4ac = 9x^2y^2 > 0$ for $xy \neq 0$ so the PDE is hyperbolic. For modified hyperbolic form, it is necessary to solve

$$2y^2r_x^2 - 5xyr_xr_y + 2x^2r_y^2 = 0, \quad 2y^2s_x^2 - 5xys_xs_y + 2x^2s_y^2 = 0,$$

noting that they factor

$$(2yr_x - xr_y)(yr_x - 2xr_y) = 0, \quad (2ys_x - xs_y)(ys_x - 2xs_y) = 0.$$

Thus, solving

$$2yr_x - xr_y = 0, \quad ys_x - 2xs_y = 0.$$

gives r and s as

$$r = x^2 + 2y^2, \quad s = 2x^2 + y^2.$$

The derivatives transform as follows:

$$\begin{aligned} (i) \quad & u_x = 2xu_r + 4xu_s, \quad u_y = 4yu_r + 2yu_s, \\ (ii) \quad & u_{xx} = 4x^2u_{rr} + 16x^2u_{rs} + 16x^2u_{ss} + 2u_r + 4u_s, \\ (iii) \quad & u_{xy} = 8xyu_{rr} + 20xyu_{rs} + 8xyu_{ss}, \\ (iv) \quad & u_{yy} = 16y^2u_{rr} + 16y^2u_{rs} + 4y^2u_{ss} + 4u_r + 2u_s, \end{aligned}$$

thus transforming the PDE to modified hyperbolic form

$$u_{rs} - \frac{su_r + ru_s}{(2r - s)(2s - r)} = 0.$$

For regular hyperbolic form, it is necessary to solve

$$2y^2r_x^2 - 5xyr_xr_y + 2x^2r_y^2 = 0, \quad 2y^2s_x^2 - 5xys_xs_y + 2x^2s_y^2 = 0,$$

using the quadratic formula. Dividing by r_y^2 gives

$$2y^2 \left(\frac{r_x}{r_y} \right)^2 - 5xy \frac{r_x}{r_y} + 2x^2 = 0,$$

and solving gives

$$\frac{r_x}{r_y} = \frac{(5 \pm 3)x}{4y}.$$

Therefore, we solve the first order PDE

$$4yr_x - (5 \pm 3)xr_y = 0.$$

Using the method of characteristics gives

$$\begin{aligned} r, s &= (5 \pm 3)x^2 + 4y^2, \\ r, s &= 5x^2 + 4y^2 \pm 3x^2, \end{aligned}$$

where we choose

$$r = 5x^2 + 4y^2, \quad s = 3x^2.$$

The derivatives transform as follows:

$$\begin{aligned} (i) \quad & u_x = 10xu_r + 6xu_s, \quad u_y = 8yu_s, \\ (ii) \quad & u_{xx} = 100x^2u_{rr} + 120x^2u_{rs} + 36x^2u_{ss} + 10u_r + 6u_s, \\ (iii) \quad & u_{xy} = 80xyu_{rr} + 48xyu_{rs}, \\ (iv) \quad & u_{yy} = 64y^2u_{rr} + 8u_r, \end{aligned}$$

thus transforming the PDE to regular hyperbolic form

$$u_{rr} - u_{ss} - \frac{1}{3} \left(\frac{8}{3r - 5s} + \frac{5}{2s} \right) u_r - \frac{1}{2s} u_s = 0.$$

1(vi). The PDE to transform is

$$x^2u_{xx} - 2xu_{xy} + 5u_{yy} + xu_x = 0.$$

Here $a = x^2$, $b = -2x$ and $c = 5$ so $b^2 - 4ac = -16x^2 < 0$ for $x \neq 0$ so the PDE is elliptic.

Therefore, it is necessary to solve

$$x^2r_x^2 - 2xr_xr_y + 5r_y^2 = 0, \quad x^2s_x^2 - 2xs_xs_y + 5s_y^2 = 0,$$

using the quadratic formula. Dividing by r_y^2 gives

$$x^2 \left(\frac{r_x}{r_y} \right)^2 - 2x \frac{r_x}{r_y} + 5 = 0,$$

and solving gives

$$\frac{r_x}{r_y} = \frac{2x \pm 4xi}{2x^2} = \frac{1 \pm 2i}{x}.$$

Therefore, we solve the first order PDE

$$xr_x - (1 \pm 2i)r_y = 0.$$

Using the method of characteristics gives

$$\frac{dx}{x} = -\frac{dy}{1 \pm 2i},$$

which leads to

$$r, s = (1 \pm 2i) \ln x + y,$$

$$r, s = \ln x + y \pm 2 \ln x.$$

where we choose

$$r = \ln x + y, \quad s = 2 \ln x.$$

The derivatives transform as follows:

$$(i) \quad u_x = \frac{1}{x}u_r + \frac{2}{x}u_s, \quad u_y = u_r,$$

$$(ii) \quad u_{xx} = \frac{1}{x^2}u_{rr} + \frac{4}{x^2}u_{rs} + \frac{4}{x^2}u_{ss} - \frac{1}{x^2}u_r - \frac{2}{x^2}u_s,$$

$$(iii) \quad u_{xy} = \frac{1}{x}u_{rr} + \frac{2}{x}u_{rs},$$

$$(iv) \quad u_{yy} = u_{rr},$$

thus transforming the PDE to elliptic form

$$4u_{rr} + 4u_{ss} = 0, \quad \Rightarrow \quad u_{rr} + u_{ss} = 0.$$