

Math 4315 - PDEs Sample Test 3 Solutions

1. Determine the Fourier series for

(i)

$$f(x) = \begin{cases} 1 & \text{if } -2 \leq x < 0 \\ x+1 & \text{if } 0 \leq x \leq 2 \end{cases}$$

(ii)

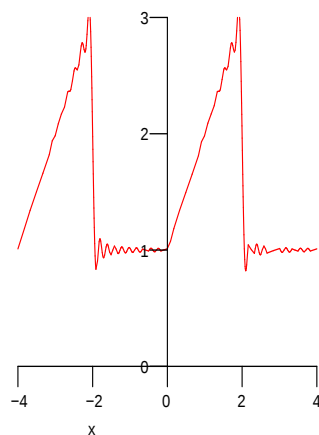
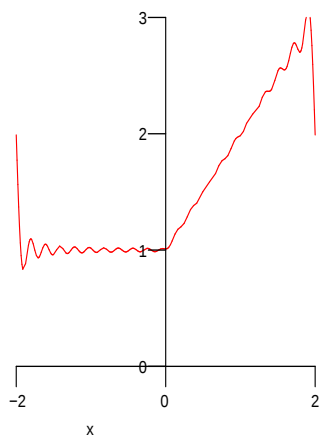
$$f(x) = \begin{cases} -x & \text{if } -1 \leq x < 0 \\ x^2 & \text{if } 0 \leq x \leq 1 \end{cases}$$

Solution 1i

$$\begin{aligned} a_0 &= \frac{1}{2} \int_{-2}^0 1 \, dx + \frac{1}{2} \int_0^2 (x+1) \, dx = 3 \\ a_n &= \frac{1}{2} \int_{-2}^0 \cos \frac{n\pi x}{2} \, dx + \frac{1}{2} \int_0^2 (x+1) \cos \frac{n\pi x}{2} \, dx = \frac{2(\cos n\pi - 1)}{n^2\pi^2} \\ b_n &= \frac{1}{2} \int_{-2}^0 \sin \frac{n\pi x}{2} \, dx + \frac{1}{2} \int_0^2 (x+1) \sin \frac{n\pi x}{2} \, dx = -\frac{2 \cos n\pi}{n\pi} \end{aligned}$$

The solution is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{i=1}^{\infty} a_n \cos \frac{n\pi x}{2} + b_n \sin \frac{n\pi x}{2} \\ &= \frac{3}{2} + \sum_{i=1}^{\infty} \frac{2(\cos n\pi - 1)}{n^2\pi^2} \cos \frac{n\pi x}{2} - \frac{2 \cos n\pi}{n\pi} \sin \frac{n\pi x}{2} \end{aligned}$$



Solution 1ii

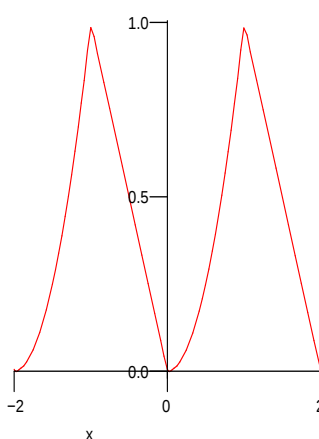
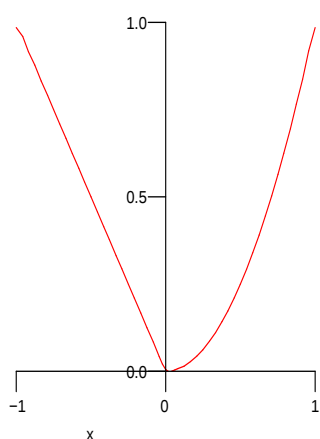
$$a_0 = \int_{-1}^0 -x \, dx + \int_0^1 x^2 \, dx = \frac{5}{6}$$

$$a_n = \int_{-1}^0 -x \cos n\pi x \, dx + \int_0^1 x^2 \cos n\pi x \, dx = \frac{3 \cos n\pi - 1}{n^2 \pi^2}$$

$$b_n = \int_{-1}^0 -x \sin n\pi x \, dx + \int_0^1 x^2 \sin n\pi x \, dx = \frac{2(\cos n\pi - 1)}{n^3 \pi^3}$$

The solution is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{i=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x \\ &= \frac{5}{12} + \sum_{i=1}^{\infty} \frac{3 \cos n\pi - 1}{n^2 \pi^2} \cos n\pi x + \frac{2(\cos n\pi - 1)}{n^3 \pi^3} \sin n\pi x. \end{aligned}$$



2. Solve

$$u_t = u_{xx}, \quad 0 < x < L,$$

subject to the initial condition and boundary conditions

$$(i) \quad u(x, 0) = 5x - x^2, \quad u(0, t) = 0, \quad u(4, t) = 4$$

$$(ii) \quad u(x, 0) = \begin{cases} x^2 + 1 & \text{if } 0 < x < 1, \\ 2(x - 2)^2 & \text{if } 1 < x < 2. \end{cases}, \quad u(0, t) = 1, \quad u(2, t) = 0$$

Solution 2i

Before we can use separation of variables, it is necessary to transform this problem to one that has fixed zero boundary conditions. If we let

$$u = v + ax + b,$$

we find that choosing $a = 1$ and $b = 0$ given the new problem to solve

$$\begin{aligned} v_t &= v_{xx}, & 0 < x < L, \\ v(x, 0) &= 4x - x^2, & v(0, t) = 0, & v(4, t) = 0. \end{aligned}$$

If we assume separable solutions in the form $v = X(x)T(t)$, then PDE separates giving

$$\frac{T'}{T} = \frac{X''}{X},$$

from which we obtain

$$T' = \lambda T, \quad X'' = \lambda X.$$

The boundary conditions become $X(0) = 0$, $X(4) = 0$. The solution of the X equation is

$$X = c_1 \sin kx + c_2 \cos kx,$$

where $\lambda = -k^2$. To satisfy both BCs we must choose $\lambda = \frac{n^2\pi^2}{16}$ and $c_2 = 0$. This then gives

$$X = c_1 \sin \frac{n\pi}{4}x.$$

Solving for T gives

$$T = c_3 e^{-\frac{n^2\pi^2}{16}t}$$

which in turn gives

$$v = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{16}t} \sin \frac{n\pi}{4}x,$$

where we have taken $b_n = c_1 c_3$. Imposing the initial condition gives

$$b_n = \frac{2}{4} \int_0^4 (4x - x^2) \sin \frac{n\pi}{4}x \, dx = \frac{64(1 - \cos n\pi)}{n^3\pi^3}.$$

This then gives the solution as

$$v = \sum_{n=1}^{\infty} \frac{64(1 - \cos n\pi)}{n^3 \pi^3} e^{-\frac{n^2 \pi^2}{16} t} \sin \frac{n\pi}{4} x,$$

and u is

$$u = x + \sum_{n=1}^{\infty} \frac{64(1 - \cos n\pi)}{n^3 \pi^3} e^{-\frac{n^2 \pi^2}{16} t} \sin \frac{n\pi}{4} x,$$

Solution 2ii

Before we can use separation of variables, it is necessary to transform this problem to one that has fixed zero boundary conditions. If we let

$$u = v + ax + b,$$

we find that choosing $a = -1/2$ and $b = 1$ given the new problem to solve

$$v_t = v_{xx}, \quad 0 < x < 2,$$

$$v(x, 0) = \begin{cases} x^2 + \frac{x}{2} & \text{if } 0 < x < 1, \\ 2x^2 - \frac{15}{2}x + 7 & \text{if } 1 < x < 2. \end{cases}, \quad v(0, t) = 0, \quad v(2, t) = 0.$$

If we assume separable solutions in the form $u = X(x)T(t)$, then PDE separates giving

$$\frac{T'}{T} = \frac{X''}{X},$$

from which we obtain

$$T' = \lambda T, \quad X'' = \lambda X.$$

The boundary conditions become $X(0) = 0$, $X(2) = 0$. The solution of the X equation is

$$X = c_1 \sin kx + c_2 \cos kx,$$

where $\lambda = -k^2$. To satisfy both BCs we must choose $\lambda = \frac{n^2 \pi^2}{4}$ and $c_2 = 0$. This then gives

$$X = c_1 \sin \frac{n\pi}{2} x.$$

Solving for T gives

$$T = c_3 e^{-\frac{n^2 \pi^2}{4} t}$$

which in turn gives

$$u = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2 \pi^2}{4} t} \sin \frac{n\pi}{2} x,$$

where we have taken $b_n = c_1 c_3$. Imposing the initial condition gives

$$\begin{aligned} b_n &= \frac{2}{2} \int_0^1 \left(x^2 - \frac{x}{2}\right) \sin \frac{n\pi}{2} x dx + \frac{2}{2} \int_1^2 \left(2x^2 - \frac{15x}{2} + 7\right) \sin \frac{n\pi}{2} x dx \\ &= \frac{(-16 - 16 \cos \frac{n\pi}{2} + 32 \cos n\pi)}{n^3 \pi^3} + \frac{24 \sin \frac{n\pi}{2}}{n^2 \pi^2}. \end{aligned}$$

This then gives v as

$$v = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{4}t} \sin \frac{n\pi}{2}x,$$

and u as

$$u = -\frac{1}{2}x + 1 + \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{4}t} \sin \frac{n\pi}{2}x,$$

3. Solve Laplace's equation

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad 0 < y < L,$$

subject to the boundary conditions

$$(i) \quad u(x, 0) = 0, \quad u(0, y) = 0, \quad u(x, 1) = x^2, \quad u(1, y) = 0,$$

$$(ii) \quad u(x, 0) = 0, \quad u(0, y) = 0, \quad u(x, 2) = 0, \quad u(2, y) = 2y - y^2.$$

Solution 3i

If we assume separable solutions of the form

$$u(x, y) = X(x)Y(y),$$

then

$$X''Y + XY'' = 0.$$

or

$$\frac{X''}{X} + \frac{Y''}{Y} = 0,$$

This gives

$$\frac{X''}{X} = \lambda, \quad \frac{Y''}{Y} = -\lambda, \quad \lambda \text{ constant.}$$

The boundary conditions become

$$X(0) = 0, \quad X(1) = 0, \quad Y(0) = 0.$$

In order to satisfy the X BCs, we need $\lambda = -k^2$ and so solving for X gives

$$X = c_1 \sin kx + c_2 \cos kx.$$

The X boundary conditions gives $k = n\pi$, $k \in \mathbb{Z}^+$ and $c_2 = 0$ so

$$X(x) = c_1 \sin n\pi x,$$

and further

$$Y(y) = c_3 \sinh n\pi y + c_4 \cosh n\pi y.$$

Since $Y(0) = 0$ this implies $c_4 = 0$ so

$$u = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi y. \quad (a_n = c_1 c_3)$$

From the last boundary condition

$$u(x, 1) = x^2 = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi,$$

If $A_n = a_n \sinh n\pi$, then

$$A_n = \frac{2}{1} \int_0^1 x^2 \sin n\pi x dx = \frac{4(\cos n\pi - 1)}{n^3 \pi^3} - 2 \frac{\cos n\pi}{n\pi}.$$

Thus, the solution is

$$u(x, y) = \sum_{n=1}^{\infty} \left(\frac{4(\cos n\pi - 1)}{n^3 \pi^3} - 2 \frac{\cos n\pi}{n\pi} \right) \sin n\pi x \frac{\sinh n\pi y}{\sinh n\pi}.$$

Solution 3ii

If we assume separable solutions of the form

$$u(x, y) = X(x)Y(y),$$

then

$$X''Y + XY'' = 0.$$

or

$$\frac{X''}{X} + \frac{Y''}{Y} = 0,$$

This gives

$$\frac{X''}{X} = \lambda, \quad \frac{Y''}{Y} = -\lambda, \quad \lambda \text{ constant.}$$

The boundary conditions become

$$X(0) = 0, \quad Y(0) = 0, \quad Y(2) = 0.$$

In order to satisfy the Y BCs, we need $\lambda = k^2$ and so solving for Y gives

$$Y = c_1 \sin ky + c_2 \cos ky.$$

The Y boundary conditions gives $k = \frac{n\pi}{2}$, $k \in \mathbb{Z}^+$ and $c_2 = 0$ so

$$Y(y) = c_1 \sin \frac{n\pi}{2} y,$$

and further

$$X(x) = c_3 \sinh \frac{n\pi}{2} x + c_4 \cosh \frac{n\pi}{2} x.$$

Since $X(0) = 0$ this implies $c_4 = 0$ so

$$u = \sum_{n=1}^{\infty} a_n \sinh \frac{n\pi}{2} x \sin \frac{n\pi}{2} y. \quad (a_n = c_1 c_3)$$

From the last boundary condition

$$u(2, y) = 2y - y^2 = \sum_{n=1}^{\infty} a_n \sinh n\pi \sin \frac{n\pi}{2} y,$$

If $A_n = a_n \sinh n\pi$, then

$$A_n = \frac{2}{2} \int_0^2 (2y - y^2) \sin \frac{n\pi}{2} y dy = \frac{16(1 - \cos n\pi)}{n^3 \pi^3}.$$

Thus, the solution is

$$u(x, y) = \sum_{n=1}^{\infty} \frac{16(1 - \cos n\pi)}{n^3 \pi^3} \frac{\sinh \frac{n\pi}{2} x}{\sinh n\pi} \sin \frac{n\pi}{2} y.$$