PDE Review

1 Jacobians and the Chain Rule

1.1 Chain rule

Consider

$$u = x^2 y \tag{1}$$

and the change of variables

$$x = r^2 + s^2, \qquad y = rs. \tag{2}$$

Eliminating x and y gives

$$u = rs\left(r^2 + s^2\right)^2. \tag{3}$$

If we wish to calculate partial derivatives with respect to r and s then we would simply calculate partial derivatives obtaining

$$u_r = 4r^2 s\left(r^2 + s^2\right) + s\left(r^2 + s^2\right)^2, \tag{4a}$$

$$u_s = 4rs^2\left(r^2 + s^2\right) + r\left(r^2 + s^2\right)^2. \tag{4b}$$

In calculus III, we were introduced to the chain rule for functions of two variables, namely

$$u_r = u_x x_r + u_y y_r, \tag{5a}$$

$$u_s = u_x x_s + u_y y_s. \tag{5b}$$

For the preceding example

$$\begin{aligned} u_x &= 2xy, & u_y &= x^2, \\ x_r &= 2r, & x_s &= 2s, \\ y_r &= s, & y_s &= r. \end{aligned}$$

and from (5) gives

$$\begin{aligned}
u_r &= 2xy2r + x^2 s \\
&= 4r^2 s\left(r^2+s^2\right) + s\left(r^2+s^2\right)^2, \\
u_s &= 2xy2s + x^2 r, \\
&= 4rs^2\left(r^2+s^2\right) + r\left(r^2+s^2\right)^2,
\end{aligned}$$

giving exactly (4). For a second example, consider $u = u(x, y)$ and the change of variables

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{6}$$

If we were asked to find u_x and u_y in terms of u_r and u_s, then we would use

$$u_x = u_r r_x + u_s s_x, \tag{7a}$$

$$u_y = u_r r_y + u_s s_y. \tag{7b}$$

To use (7), it is necessary to solve (6) for r and θ, *i.e.*

$$r = \sqrt{x^2+y^2}, \qquad \theta = \tan^{-1}\frac{y}{x}, \tag{8}$$

and then from (7) we obtain (using r_x, r_y, s_x and s_y where appropriate)

$$u_x = \frac{x}{\sqrt{x^2+y^2}}u_r - \frac{y}{x^2+y^2}u_s, \tag{9a}$$

$$u_y = \frac{y}{\sqrt{x^2+y^2}}u_r + \frac{x}{x^2+y^2}u_s. \tag{9b}$$

However, we note we have a problem! The right hand side of (9) involves x, y, r and s – all the variables. Therefore, we must eliminate x and y from (9) using (6). In doing so, we obtain

$$u_x = \cos\theta u_r - \frac{\sin\theta}{r}u_s \tag{10a}$$

$$u_y = \sin u_r + \frac{\cos\theta}{r}u_s. \tag{10b}$$

One might suggest that we use (5) and then solve for u_x and u_y. This would work, however, for more than two independent variables, the algebra gets very complicated. This is especially true when calculating higher order derivatives.

The natural question is − Is there a way to transform derivatives directly from $(x, y) \rightarrow (r, \theta)$? The answer is yes! We use Jacobians. Recall the Jacobian $J(u, v, x, y)$ is defined as

$$J(u,v,x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}. \tag{11}$$

In terms of Jacobians, we can write derivatives simply as

$$u_x = \frac{\partial(u,y)}{\partial(x,y)}, \qquad u_y = \frac{\partial(x,u)}{\partial(x,y)}. \tag{12}$$

and to transform from $(x, y) \rightarrow (r, \theta)$ then we just use the chain rule for Jacobians, *i.e.*

$$u_x = \frac{\partial(u,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)}, \qquad u_y = \frac{\partial(x,u)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)},$$

or

$$u_x = \frac{\partial(u,y)}{\partial(r,\theta)} \Big/ \frac{\partial(x,y)}{\partial(r,\theta)}, \qquad u_y = \frac{\partial(x,u)}{\partial(r,\theta)} \Big/ \frac{\partial(x,y)}{\partial(r,\theta)}. \tag{13}$$

Substituting (6) in (13) gives rise to (10).

If we were to transform derivatives from $(x, y, z) \rightarrow (\rho, \theta, \phi)$ where

$$x = \rho\cos\theta\sin\phi, \quad y = \rho\sin\theta\sin\phi, \quad z = \rho\cos\phi, \tag{14}$$

we would use 3×3 Jacobians. For example, consider u_x. For this we have

$$u_x = \frac{\partial(u,y,z)}{\partial(x,y,z)} \tag{15}$$

and using the chain rule, we would obtain

$$u_x = \frac{\partial(u,y,z)}{\partial(\rho,\theta,\phi)} \Big/ \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}, \tag{16}$$

where the two Jacobians are

$$\frac{\partial(u,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} u_\rho & u_\theta & u_\phi \\ \sin\theta\sin\phi & \rho\cos\theta\ sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix},$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\ sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\ sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}.$$

Simplifying each and dividing gives

$$u_x = \cos\theta\cos\phi\ u_\rho - \frac{\sin\theta}{\rho\sin\phi}\ u_\theta + \frac{\cos\theta\cos\phi}{\rho}\ u_\phi. \tag{17}$$

Similarly for u_y and u_z giving

$$u_y = \sin\theta\cos\phi\ u_\rho + \frac{\cos\theta}{\rho\sin\phi}\ u_\theta + \frac{\sin\theta\cos\phi}{\rho}\ u_\phi, \tag{18}$$

$$u_z = \cos\theta\ u_\rho - \frac{\sin\phi}{\rho}\ u_\phi. \tag{19}$$

1.2 Second Order Derivatives

In this course it will be necessary to transform 2^{nd} order derivatives. Thus, we will need a chain rule for these. Here, we will use the results from (7), whereby differentiating with respect to x gives

$$\begin{aligned} u_{xx} &= \frac{\partial}{\partial x}\left(u_x\right) \\ &= \frac{\partial}{\partial x}\left(u_r r_x + u_s s_x\right) \\ &= \frac{\partial}{\partial x} u_r r_x + u_r r_{xx} + \frac{\partial}{\partial x} u_s s_x + u_s s_{xx} \end{aligned}$$

and using the relation that

$$\frac{\partial}{\partial x} = r_x \frac{\partial}{\partial r} + s_x \frac{\partial}{\partial s}$$

gives

$$u_{xx} = r_x^2 u_{rr} + 2 r_x s_x u_{rs} + s_x^2 u_{ss} + r_{xx} u_r + s_{xx} u_s. \tag{20}$$

Similar results apply for u_{xy} and u_{yy} and are given by

$$u_{xy} = r_x r_y u_{rr} + \left(r_x s_y + r_y s_x\right) u_{rs} + s_x s_y u_{ss} + r_{xy} u_r + s_{xy} u_s, \tag{21}$$

$$u_{yy} = r_y^2 u_{rr} + 2 r_y s_y u_{rs} + s_y^2 u_{ss} + r_{yy} u_r + s_{yy} u_s. \tag{22}$$

Example 1

If

$$r = x - y, \qquad s = x + y,$$

then

$$\begin{gathered} r_x = 1, \quad r_y = -1, \quad r_{xx} = 0, \quad r_{xy} = 0, \quad r_{yy} = 0, \\ s_x = 1, \quad s_y = 1, \quad s_{xx} = 0, \quad s_{xy} = 0, \quad s_{yy} = 0, \end{gathered}$$

which gives (20)$-$(22) as

$$\begin{aligned} u_{xx} &= u_{rr} + 2u_{rs} + u_{ss}, \\ u_{xy} &= -u_{rr} + u_{ss}, \\ u_{yy} &= u_{rr} - 2u_{rs} + u_{ss}. \end{aligned}$$

Example 2

In this example, consider

$$x = r^2 - s^2, \quad y = 2rs. \tag{23}$$

In order to use the chain rules (20)−(22) we would need to solve (23) for r and s (yuk!). However, it is easier to use Jacobians. From (13) (with θ replaced with s) we obtain

$$u_x = \frac{1}{2}\frac{ru_r - su_s}{r^2+s^2}, \qquad u_y = \frac{1}{2}\frac{su_r + ru_s}{r^2+s^2}.$$

Identifying the operators

$$\frac{\partial}{\partial x} = \frac{1}{2}\frac{r}{r^2+s^2}\frac{\partial}{\partial r} - \frac{1}{2}\frac{s}{r^2+s^2}\frac{\partial}{\partial s}, \qquad \frac{\partial}{\partial y} = \frac{1}{2}\frac{s}{r^2+s^2}\frac{\partial}{\partial r} + \frac{1}{2}\frac{r}{r^2+s^2}\frac{\partial}{\partial s}$$

then to find u_{xx}, it is sufficient to apply the first operator twice, *i.e.*

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{1}{4}\left(\frac{r}{r^2+s^2}\frac{\partial}{\partial r} - \frac{s}{r^2+s^2}\frac{\partial}{\partial s}\right)\left(\frac{r}{r^2+s^2}\frac{\partial}{\partial r} - \frac{s}{r^2+s^2}\frac{\partial}{\partial s}\right)u,\\
&= \frac{1}{4}\left(\frac{r}{r^2+s^2}\frac{\partial}{\partial r} - \frac{s}{r^2+s^2}\frac{\partial}{\partial s}\right)\left(\frac{r}{r^2+s^2}\frac{\partial u}{\partial r} - \frac{s}{r^2+s^2}\frac{\partial u}{\partial s}\right),\\
&= \frac{r^2u_{rr} - 2rsu_{rs} + s^2u_{ss}}{4(r^2+s^2)^2} - \frac{r(r^2-3s^2)u_r - s(3r^2-s^2)u_s}{4(r^2+s^2)^3}.
\end{aligned} \tag{24}$$

Similar results apply for u_{xy} and u_{yy}.

2 Differential Equations

2.1 First Order Equations

Ordinary differential equations of the form

$$y' = F(x,y) \tag{25}$$

are called first order ordinary differential equations. There are a variety of techniques to solve these type of equations and main methods are:

(i) separable
(ii) linear
(iii) Bernoulli
(iv) Ricatti
(v) homogeneous
(vi) linear fractional
(vii) exact
(viii) Legendre transformations

2.1.1 Separable Equations

A separable first order differential equation has the form:

$$\frac{dy}{dx} = f(x)g(y) \tag{26}$$

The general solution is found by separating the differential equation and integrating including a single constant of integration, *i.e.*

$$\int \frac{1}{g(y)} dy = \int f(x)dx + c.$$

For example, to solve

$$y' = xy + x + 2y + 2,$$

it is necessary to rewrite it as

$$y' = (x + 2)(y + 1).$$

Separating variables and writing in integral form gives

$$\int \frac{dy}{y+1} = \int x + 2\, dx,$$

and integrating yields

$$\ln|y + 1| = \frac{x^2}{2} + 2x + c.$$

Letting $c = \ln(k)$ and solving for y gives:

$$y = k \cdot e^{x^2/2+2x} - 1.$$

2.1.2 Linear Equations

Equations of this type are in the form

$$\frac{dy}{dx} + p(x)y = q(x) \tag{27}$$

To solve this, we introduce the integrating factor

$$\mu = e^{\int p(x)dx} \tag{28}$$

This is designed so that when both sides of (27) are multiplied by μ, the left side (27) is a derivative of a product, that is, it becomes:

$$\mu\left(\frac{dy}{dx} + py\right) = \frac{d}{dx}(\mu y),$$

and then (27) can be integrated. For example, if

$$\frac{dy}{dx} - \frac{2y}{x} = 2x^3 - 1, \tag{29}$$

then $p(x) = -\frac{2}{x}$, so that

$$\mu = e^{-2\int \frac{dx}{x}} = e^{-2\ln x} = \frac{1}{x^2}.$$

On multiplying (29) by μ gives

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = 2x - \frac{1}{x^2},$$

which simplifies to

$$\frac{d}{dx}\left(\frac{1}{x^2} \cdot y\right) = 2x - \frac{1}{x^2}.$$

Integrating gives

$$\frac{1}{x^2} \cdot y = x^2 + \frac{1}{x} + c,$$

and solving for y gives

$$y = x^4 + x + cx^2.$$

2.1.3 Bernoulli

Equations of the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n \qquad (n \neq 0, 1) \tag{30}$$

are called Bernoulli equations. Dividing both sides of (30) by y^n gives

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{p(x)}{y^{n-1}} = q(x) \tag{31}$$

Let $v = \frac{1}{y^{n-1}}$, then $\frac{dv}{dx} = (1-n)\frac{1}{y^n}\frac{dy}{dx}$ or $\frac{1}{(1-n)}\frac{dv}{dx} = \frac{1}{y^n}\frac{dy}{dx}$. Upon making this substitution into (31) gives

$$\frac{1}{1-n}\frac{dv}{dx} + p(x)v = q(x)$$

which is linear. So Bernoulli equations can be reduced to linear equations.

Example

$$\frac{dy}{dx} - \frac{y}{2x} = y^3.$$

This is an example where $n = 3$. Putting this into standard form gives

$$\frac{1}{y^3}\frac{dy}{dx} - \frac{1}{2x}\frac{1}{y^2} = 1 \tag{32}$$

Letting $v = \frac{1}{y^2}$, then $\frac{dv}{dx} = \frac{-2}{y^3}\frac{dy}{dx}$, and (32) is transformed to

$$-\frac{1}{2}\frac{dv}{dx} - \frac{1}{2x}v = 1,$$

or

$$\frac{dv}{dx} + \frac{v}{x} = -2.$$

As this is linear, then $p(x) = \frac{1}{x}$, and the integrating factor for this is x, so that

$$\frac{d}{dx}(xv) = x\frac{dv}{dx} + v = -2x,$$

and thus

$$xv = -x^2 + c,$$

or

$$v = -x + \frac{c}{x}.$$

So that

$$\frac{1}{y^2} = -x + \frac{c}{x},$$

or

$$y = \frac{\pm 1}{\sqrt{-x + \frac{c}{x}}}.$$

2.1.4 Ricatti Equations

Ricatti equations have the form:

$$\frac{dy}{dx} = a(x)y^2 + b(x)y + c(x). \tag{33}$$

To find a general solution to this requires having one solution first. Given this solution, it is possible to change equation (33) to a linear equation. If we let

$$y = y_0 + \frac{1}{u},$$

where y_0 is a solution to (33), then (33) is transformed to the linear equation

$$u' = -\left(2a(x)y_0 + b(x)\right)u - a(x),$$

which is linear. To illustrate, we consider the following example

$$\frac{dy}{dx} = -\frac{y^2}{x^2} + \frac{y}{x} + 1. \tag{34}$$

Since $y_0 = x$ is a solution to this equation, let $y = x + \frac{1}{u}$, and (34) becomes

$$1 - \frac{u'}{u^2} = -\frac{1}{x^2}\left(x^2 + 2\frac{x}{u} + \frac{1}{u^2}\right) + \frac{1}{x}\left(x + \frac{1}{u}\right) + 1.$$

Simplifying gives

$$-\frac{u'}{u^2} = -\frac{1}{xu} - \frac{1}{x^2u^2},$$

and multiplying by $-u^2$ and rearranging gives rise to the linear equation

$$u' - \frac{1}{x}u = \frac{1}{x^2}. \tag{35}$$

Here $p(x) = -\frac{1}{x}$ so this has the integrating factor $\mu = e^{\int \frac{-dx}{x}} = \frac{1}{x}$, so (35) becomes

$$\frac{u'}{x} - \frac{u}{x^2} = \frac{d}{dx}\left(\frac{u}{x}\right) = \frac{1}{x^3},$$

and upon integration gives

$$\frac{u}{x} = -\frac{1}{2x^2} + c,$$

or

$$u = cx - \frac{1}{2x}.$$

Since $y = x + \frac{1}{u}$, this gives y as

$$y = x + \frac{1}{cx - \frac{1}{2x}}.$$

2.1.5 Homogeneous Equations

Equations of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \tag{36}$$

are called homogeneous equations. Substituting $y = xu$ will yield the equation

$$x\frac{du}{dx} + u = F(u).$$

which separates to

$$\frac{du}{F(u) - u} = \frac{dx}{x}.$$

Consider the previous example,

$$\frac{dy}{dx} = -\frac{y^2}{x^2} + \frac{y}{x} + 1. \tag{37}$$

If we let $y = xu$, then (37) becomes

$$\frac{d(xu)}{dx} = -u^2 + u + 1,$$

or

$$x\frac{du}{dx} + u = -u^2 + u + 1$$

and simplifying and separating gives

$$\frac{du}{1-u^2} = \frac{dx}{x}.$$

Integrating gives

$$\frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| = \ln|x| + \ln c,$$

or

$$\frac{u+1}{u-1} = cx^2$$

Since $y = xu$ this gives

$$\frac{\frac{y}{x}+1}{\frac{y}{x}-1} = cx^2$$

or

$$\frac{y+x}{y-x} = cx^2$$

solving for y leads to the solution

$$y = \frac{cx^3 + x}{cx^2 - 1}.$$

2.1.6 Linear Fractional

Equations that have the form

$$\frac{dy}{dx} = \frac{ax + by + e}{cx + dy + f}, \tag{38}$$

are called linear fractional. Under a change of variables,

$$x = \bar{x} + \alpha, \quad y = \bar{y} + \beta,$$

we can change equation (38) to one that is either homogeneous ($ad - bc \neq 0$) or to one that is separable ($ad - bc = 0$). The following examples illustrate. Consider

$$\frac{dy}{dx} = \frac{2x - 3y + 8}{3x - 2y + 7}. \tag{39}$$

If we let

$$x = \bar{x} + \alpha, \;\; y = \bar{y} + \beta,$$

then (39) becomes

$$\frac{d\bar{y}}{d\bar{x}} = \frac{2\bar{x} - 3\bar{y} + 2\alpha - 3\beta + 8}{3\bar{x} - 2\bar{y} + 3\alpha - 2\beta + 7}.$$

Choosing

$$2\alpha - 3\beta + 8 = 0, \;\; 3\alpha - 2\beta + 7 = 0, \tag{40}$$

leads to

$$\frac{d\bar{y}}{d\bar{x}} = \frac{2\bar{x} - 3\bar{y}}{3\bar{x} - 2\bar{y}}, \tag{41}$$

a homogeneous equation. The natural question is, "does (40) have a solution?" In this case, it does and we can find that the solution is $\alpha = -1$ and $\beta = 2$. The solution of (41) is

$$\bar{x}^2 - 3\bar{x}\bar{y} + \bar{y}^2 = c \tag{42}$$

and from our change of variables ($x = \bar{x} - 1, \;\; y = \bar{y} + 2$) we find the solution to (39) is

$$(x+1)^2 - 3(x+1)(y-2) + (y-2)^2 = c. \tag{43}$$

As a second example, consider

$$\frac{dy}{dx} = \frac{4x - 2y + 8}{2x - y + 7}. \tag{44}$$

If we let

$$x = \bar{x} + \alpha, \;\; y = \bar{y} + \beta,$$

then

$$\frac{d\bar{y}}{d\bar{x}} = \frac{4\bar{x} - 2\bar{y} + 4\alpha - 2\beta + 8}{2\bar{x} - \bar{y} + 2\alpha - \beta + 7}.$$

We choose

$$4\alpha - 2\beta + 8 = 0, \;\; 2\alpha - \beta + 7 = 0,$$

but this has no solution! However, if we let $u = 2x - y$, then (44) becomes

$$\frac{du}{dx} = \frac{6}{u + 7},$$

which is separable! Its solution is given by

$$u^2 + 14u = 12x + c,$$

and upon back substitution, we obtain the solution of (44) as

$$(2x - y)^2 + 14(2x - y) = 12x + c,$$

2.1.7 Exact Equations

An ordinary differential equation of the form

$$\frac{dy}{dx} = F(x, y), \tag{45}$$

has the alternate form

$$M(x, y)dx + N(x, y)dy = 0. \tag{46}$$

If M and N have continuous partial derivatives of first order in some region R and

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

then the ODE (45) or (46) is said to be "exact" and can be integrated by setting

$$\frac{\partial \phi}{\partial x} = M, \quad \text{and} \quad \frac{\partial \phi}{\partial y} = N.$$

For example, consider the differential equation

$$\frac{dy}{dx} = -\frac{2xy}{x^2 + y^2}, \tag{47}$$

which can be written as

$$2xy\, dx + (x^2 + y^2)dy = 0. \tag{48}$$

If we identify that M and N are

$$M = 2xy \quad \text{and} \quad N = x^2 + y^2,$$

so

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x,$$

so $2xy\, dx + (x^2 + y^2)dy = 0$ is exact. Setting

$$\frac{\partial \phi}{\partial x} = 2xy, \quad \text{and} \quad \frac{\partial \phi}{\partial y} = x^2 + y^2.$$

then from the first of these equations upon integrating we get

$$\phi = x^2 y + g(y),$$

taking the partial of this with respect to y gives

$$\frac{\partial \phi}{\partial y} = x^2 + g'(y).$$

Comparing this to $\frac{\partial \phi}{\partial y} = x^2 + y^2$ gives that

$$g'(y) = y^2,$$

so

$$g(y) = \frac{y^3}{3} + c,$$

so that

$$\phi = x^2 y + \frac{y^3}{3} + c.$$

From Cal III we know that

$$d\phi = \phi_x dx + \phi_y dy,$$

but in this case this is

$$d\phi = 2xydx + (x^2 + y^2)dy = 0$$

so ϕ is a constant. Thus we have as solutions to

$$2xy\,dx + (x^2 + y^2)dy = 0$$

$$\phi = k \quad \text{or} \quad x^2 y + \frac{y^3}{3} + c = k,$$

and absorbing the constant k into c gives

$$x^2 y + \frac{y^3}{3} + c = 0$$

as the set of possible solutions to (47).

2.1.8 Legendre Transformations

Sometimes it is necessary to solve more general equations of the form

$$F(x, y, y') = 0, \tag{49}$$

say, for example

$$y'^2 - xy' + 3y = 0. \tag{50}$$

One possibility is to introduce a contact transformation that enables one to solve a given equation. Contact transformations, in general, are of the form

$$x = F(X, Y, Y'), \quad y = G(X, Y, Y'), \quad y' = H(X, Y, Y'), \tag{51}$$

with the contact condition that

$$\frac{G_X + G_Y y' + G_{Y'} Y''}{F_X + F_Y y' + F_{Y'} Y''} = H.$$

One such contact transformation is called a Legendre transformation and is given by

$$x = \frac{dY}{dX}, \quad y = X\frac{dY}{dX} - Y, \quad y' = X. \tag{52}$$

One can verify that

$$\frac{dy}{dx} = \frac{\frac{d}{dX}\left(X\frac{dY}{dX} - Y\right)}{\frac{d}{dX}\left(\frac{dY}{dX}\right)} = \frac{X\frac{d^2Y}{dX^2}}{\frac{d^2Y}{dX^2}} = X.$$

Substitution of (52) in (50) gives

$$2X\frac{dY}{dX} - 3Y + X^2 = 0 \tag{53}$$

a linear ODE! Solving gives

$$Y = CX^{\frac{3}{2}} - X^2. \tag{54}$$

Substituting (54) back into (52) gives

$$x = \frac{3}{2}cX^{\frac{1}{2}} - 2X, \quad y = \frac{1}{2}cX^{\frac{3}{2}} - X^2. \tag{55}$$

Solving the first of (55) for $X^{\frac{1}{2}}$ gives

$$X^{\frac{1}{2}} = \frac{3c \pm \sqrt{9c^2 + 32x}}{8}, \tag{56}$$

and from the second of (55) gives

$$y = \frac{c}{2}\left(\frac{3c \pm \sqrt{9c^2 + 32x}}{8}\right)^3 - \left(\frac{3c \pm \sqrt{9c^2 + 32x}}{8}\right)^4,$$

the exact solution of (50).

2.2 Linear Systems

A linear system of equations

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy, \tag{57}$$

can be can be written as a matrix ODE

$$\frac{d\bar{x}}{dt} = A\bar{x} \tag{58}$$

where $\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\bar{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If we consider solutions of the form

$$\bar{x} = \bar{c}\mathrm{e}^{\lambda t},$$

then after substitution into (58) we obtain

$$\lambda\bar{c}\,\mathrm{e}^{\lambda t} = A\,\bar{c}\,\mathrm{e}^{\lambda t}$$

from which we deduce

$$(\lambda I - A)\,\bar{c} = 0. \tag{59}$$

In order to have nontrivial solutions $\bar{c}$, we require that

$$|\lambda I - A| = 0. \tag{60}$$

This is the eigenvalue-eigenvector problem. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then (60) becomes

$$\lambda^2 - (a+d)\lambda + a\,d - b\,c = 0,$$

from which we have three cases:

(i) two distinct eigenvalues

(ii) two repeated eigenvalues,

(iii) two complex eigenvalues.

Here we consider an example of the first, two distinct eigenvalues. If

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \bar{x} \tag{61}$$

then the characteristic equation is

$$\begin{vmatrix} \lambda - 1 & -1 \\ -2 & \lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2) = 0,$$

from which we obtain the eigenvalues $\lambda = -1$ and $\lambda = 2$.

Case 1: $\lambda = -1$

From (59) we have

$$\begin{pmatrix} -2 & -1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $2c_1 + c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Case 2: $\lambda = 2$

From (59) we have

$$\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $c_1 - c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution to (61) is then given by

$$\bar{x} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \mathrm{e}^{-\mathrm{t}} + \mathrm{c}_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{2\mathrm{t}}.$$

2.2.1 Alternate Form

Sometimes a system of ODEs can be written as

$$\frac{dt}{P(t,x,y)} = \frac{dx}{Q(t,x,y)} = \frac{dy}{R(t,x,y)}. \tag{62}$$

This is similar to the alternate form for a single ODE

$$\frac{dy}{dx} = F(x,y) \text{ or } M(x,y)dx + N(x,y)dy = 0.$$

One could write (62) in terms of the usual system

$$\frac{dx}{dt} = \frac{Q}{P}, \ \frac{dy}{dt} = \frac{R}{P},$$

and determine whether its linear or nonlinear and proceed as above but sometimes its not possible nor desirable.

Example 1

Consider the following

$$\frac{dx}{x} = \frac{dy}{2y} = \frac{du}{3u}.$$

Here, it is easier to pick them in pairs, say for example

$$\frac{dx}{x} = \frac{dy}{2y}, \quad \frac{dx}{x} = \frac{du}{3u}.$$

Each are easily solved giving rise to

$$\frac{y}{x^2} = c_1, \quad \frac{u}{x^3} = c_2.$$

Example 2

Consider

$$\frac{dx}{u-x} = \frac{dy}{2x} = \frac{du}{u-x}. \tag{63}$$

Here we need to be somewhat choosy in how we pick our pairs as not all pairs will work (*i.e.* a pair with only two variables). The choice here is the first and third

$$\frac{dx}{u-x} = \frac{du}{u-x},$$

as this simplifies to

$$dx = du,$$

which integrate to $u = x + c_1$. With this we substitute into the original system and obtain

$$\frac{dx}{c_1} = \frac{dy}{2x} = \frac{dx}{c_1}.$$

noting that we now in fact have only a single pair

$$\frac{dx}{c_1} = \frac{dy}{2x}.$$

Upon integration, we obtain

$$x^2 = c_1 y + c_2,$$

and using c_1 obtained previously, we get

$$x^2 = (u-x)y + c_2.$$

Example 3

Consider

$$\frac{dx}{x} = \frac{dy}{x+y} = \frac{dz}{x+y+z}. \tag{64}$$

Again, choose wisely. Here we choose the first pair

$$\frac{dx}{x} = \frac{dy}{x+y}, \quad \text{or} \quad \frac{dy}{dx} = \frac{x+y}{x}$$

which we find as its solution

$$y = x \ln|x| + c_1 x.$$

Eliminating y in the first and third pairing in (64) gives

$$\frac{dx}{x} = \frac{dz}{x + x\ln|x| + c_1 x + z}.$$

or

$$\frac{dz}{dx} = \frac{x + x\ln|x| + c_1 x + z}{x}.$$

which is linear in z. Integrating gives

$$\frac{z}{x} = \ln|\ln|x|| + (c_1+1)\ln|x| + c_2,$$

and eliminating c_2 gives

$$\frac{z}{x} = \ln(\ln|x|) + (y - x\ln|x| + 1)\ln|x| + c_2.$$

Example 4

Consider

$$\frac{dx}{y+z} = \frac{dy}{y} = \frac{dz}{x-y}. \tag{65}$$

Here it is impossible to choose a pair that only involves 2 variables so we need to be very clever. Consider the first and second as a pair and the second and third terms as a pair and re-write as

$$\frac{dx}{dy} = \frac{y+z}{y}, \quad \frac{dz}{dy} = \frac{x-y}{y} \tag{66}$$

now here's the clever part, add and subtract the two ODEs in (66)

$$\frac{d(x+z)}{dy} = \frac{x+z}{y}, \quad \frac{d(x-z)}{dy} = \frac{2y-x+z}{y}. \tag{67}$$

If we let $u = x+z$ and $v = x-z$, then (67) becomes

$$\frac{du}{dy} = \frac{u}{y}, \quad \frac{dv}{dy} = \frac{2y-v}{y},$$

from which we find the solution

$$\frac{u}{y} = c_1, \quad yv = y^2 + c_2,$$

or, in term of the original variables

$$\frac{x+z}{y} = c_1, \quad (x-z) - y^2 = c_2.$$

Example 5

Consider

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{1} = \frac{dp}{2p} = \frac{dq}{2q}. \tag{68}$$

Here, there a 5 independent variables x, y, u, p, and q. Again, we pick in pairs. First we pick only the first two in (68)

$$\frac{dx}{x} = \frac{dy}{y}, \tag{69}$$

and obtain the solution $y = c_1 x$. Using this in (68) gives

$$\frac{dx}{x} = \frac{c_1 dx}{c_1 x} = \frac{du}{1} = \frac{dp}{2p} = \frac{dq}{2q}, \tag{70}$$

noting the first two terms in (70) are identical (after cancellation) and thus we really only have

$$\frac{dx}{x} = \frac{du}{1} = \frac{dp}{2p} = \frac{dq}{2q}, \tag{71}$$

Now we pick another pair $-$ first and second in (71) so

$$\frac{dx}{x} = \frac{du}{1},$$

so

$$u - \ln|x| = c_2.$$

The first and third in (71) integrates to

$$\frac{p}{x^2} = c_3,$$

and the first and forth in (71) integrates to

$$\frac{q}{x^2} = c_4.$$

Thus, the solution to the system (68) is

$$\frac{y}{x} = c_1, \quad u - \ln|x| = c_2, \quad \frac{p}{x^2} = c_3, \quad \frac{q}{x^2} = c_4. \tag{72}$$

Exercises

1. Calculate the first order derivatives u_x and u_y and second derivatives u_{xx}, u_{xy} and u_{yy} for the following change of coordinates (use Jacobian's on the last two)

$$\begin{aligned}(i) \quad & r = 2x - y, \quad s = x + y, \\ (ii) \quad & r = x\,\mathrm{e}^{y}, \quad s = x\,\mathrm{e}^{-y}, \\ (iii) \quad & x = r - s, \quad y = r + s, \\ (iv) \quad & x = r\,cos\,\theta, \quad y = r\,sin\,\theta,\end{aligned}$$

2. Derive the chain rule for the first and second order derivatives

$$u_x, \quad u_y, \quad u_z, \quad u_{xx}, \quad u_{xy}, \quad u_{xz}, \quad u_{yy}, \quad u_{yz}, \quad u_{zz}.$$

under the change of coordinates

$$x = r + s + t, \quad y = r - s + t, \quad z = r - s - t.$$

Hint: Use Jacobian's to derive the first order chain rule and then apply chain rule twice.

3. Solve the following first order ordinary differential equations

$$(i) \quad xy' = 3y + x^2, \qquad (ii) \quad xy' + y = x^2y^2,$$

$$(iii) \quad \tfrac{dy}{dx} = \tfrac{y^2 - 3x^2y}{x^3 - 2xy}, \qquad (iv) \quad x^2y' = x^2y^2 + xy - 3,$$

4. Solve the following systems of ODEs

$$(i) \quad \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}, \qquad (ii) \quad \frac{dx}{y} = -\frac{dy}{x} = \frac{dz}{z},$$
$$(iii) \quad \frac{dx}{y} = \frac{dy}{x - z} = \frac{dz}{y},$$

5. Derive the chain rule for first and second order derivatives when the following change of variables is made

$$\begin{aligned} x_1 &= x_1(r_1, r_2, \cdots r_n), \\ x_2 &= x_2(r_1, r_2, \cdots r_n), \\ &\vdots \\ x_n &= x_n(r_1, r_2, \cdots r_n). \end{aligned}$$

6. Solve the following system of ODEs (for some integer $n \geq 2$)

$$\frac{dx_1}{x_1} = \frac{dx_2}{x_2} = \cdots = \frac{dx_n}{x_n}.$$

7. Consider the following system of ODEs

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}.$$

Show that for some real constants a, b and c

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} = \frac{d\,(ax+by+cz)}{aP+bQ+cR}.$$