Chapter 3. Second Order Linear PDEs

3.1 Introduction

The general class of second order linear PDEs are of the form:

$$\begin{aligned} a(x,y)u_{xx} + b(x,y)u_{xy} &+ c(x,y)u_{yy} \\ + d(x,y)u_x &+ e(x,y)u_y + f(x,y)u = g(x,y). \end{aligned} \tag{3.1}$$

The three PDEs that lie at the cornerstone of applied mathematics are: the heat equation, the wave equation and Laplace's equation,*i.e.*

$$\begin{aligned} (i) \quad u_t = u_{xx}, &\quad \text{the heat equation} \\ (ii) \quad u_{tt} = u_{xx}, &\quad \text{the wave equation} \\ (iii) \quad u_{xx} + u_{yy} = 0, &\quad \text{Laplace's equation} \end{aligned}$$

or, using the same independent variables, x and y

$$(i) \quad u_{xx} - u_y = 0, \quad \text{the heat equation} \tag{3.3a}$$

$$(ii) \quad u_{xx} - u_{yy} = 0, \quad \text{the wave equation} \tag{3.3b}$$

$$(iii) \quad u_{xx} + u_{yy} = 0. \quad \text{Laplace's equation} \tag{3.3c}$$

Analogous to characterizing quadratic equations

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

as either hyperbolic, parabolic or elliptic determined by

$$\begin{aligned} b^2 - 4ac > 0, &\quad \text{hyperbolic,} \\ b^2 - 4ac = 0, &\quad \text{parabolic,} \\ b^2 - 4ac < 0, &\quad \text{elliptic,} \end{aligned}$$

we do the same for PDEs. So, for the heat equation $a = 1,\ b = 0,\ c = 0$ so $b^2 - 4ac = 0$ and so the heat equation is parabolic. Similarly, the wave equation is hyperbolic and Laplace's equation is elliptic. This leads to a natural question. Is it possible to transform one PDE to another where the new PDE is simpler? Namely, under a change of variable

$$r = r(x, y), \quad s = s(x, y),$$

can we transform to one of the following *canonical* forms:

$$u_{rr} - u_{ss} + \text{l.o.t.s.} = 0, \quad \text{hyperbolic}, \tag{3.5a}$$

$$u_{ss} + \text{l.o.t.s.} = 0, \quad \text{parabolic}, \tag{3.5b}$$

$$u_{rr} + u_{ss} + \text{l.o.t.s.} = 0, \quad \text{elliptic}, \tag{3.5c}$$

where the term "l.o.t.s" stands for lower order terms. For example, consider the PDE

$$2u_{xx} - 2u_{xy} + 5u_{yy} = 0. \tag{3.6}$$

This equation is elliptic since the elliptic $b^2 - 4ac = 4 - 40 = -36 < 0$. If we introduce new coordinates,

$$r = 2x + y, \quad s = x - y,$$

then by a change of variable using the chain rule

$$\begin{aligned}
u_{xx} &= u_{rr}r_x^2 + 2u_{rs}r_x s_x + u_{ss}s_x^2 + u_r r_{xx} + u_s s_{xx}, \\
u_{xy} &= u_{rr}r_x r_y + u_{rs}(r_x s_y + r_y s_x) + u_{ss}s_x s_y + u_r r_{xy} + u_s s_{xy}, \\
u_{yy} &= u_{rr}r_y^2 + 2u_{rs}r_y s_y + u_{ss}s_y^2 + u_r r_{yy} + u_s s_{yy},
\end{aligned}$$

gives

$$\begin{aligned}
u_{xx} &= 4u_{rr} + 4u_{rs} + u_{ss}, \\
u_{xy} &= 2u_{rr} - u_{rs} - u_{ss}, \\
u_{yy} &= u_{rr} - 2u_{rs} + u_{ss}.
\end{aligned}$$

Under (3.7), equation (3.6) becomes

$$u_{rr} + u_{ss} = 0,$$

which is Laplace's equation (also elliptic). Before we consider transformations for PDEs in general, it is important to determine whether the equation type could change under transformation. Consider the general class of PDEs

$$au_{xx} + bu_{xy} + cu_{yy} = 0 \tag{3.7}$$

where a, b, and c are functions of x and y and noting that we have suppressed the lower terms as they will not affect the type. Under a change of variable $(x, y) \rightarrow (r, s)$ with the change of variable formulas (3.7) gives

$$\begin{aligned} & a\left(u_{rr}r_x^2 + 2u_{rs}r_xs_x + u_{ss}s_x^2 + u_rr_{xx} + u_ss_{xx}\right) \\ + \quad & b\left(u_{rr}r_xr_y + u_{rs}(r_xs_y + r_ys_x) + u_{ss}s_xs_y + u_rr_{xy} + u_ss_{xy}\right) \\ + \quad & c\left(u_{yy} + u_{rr}r_y^2 + 2u_{rs}r_ys_y + u_{ss}s_y^2 + u_rr_{yy} + u_ss_{yy}\right) = 0 \end{aligned} \tag{3.8}$$

Rearranging (3.8), and again neglecting lower order terms, gives

$$\begin{aligned} (ar_x^2 + br_xr_y + cr_y^2)u_{rr} \quad + & \quad (2ar_xs_x + b(r_xs_y + r_ys_x) + 2cr_ys_y)u_{rs} \\ + & \quad (as_x^2 + bs_xs_y + cs_y^2)u_{ss} = 0. \end{aligned} \tag{3.9}$$

Setting

$$\begin{aligned} A &= ar_x^2 + br_xr_y + cr_y^2, \\ B &= 2ar_xs_x + b(r_xs_y + r_ys_x) + 2cr_ys_y, \\ C &= as_x^2 + bs_xs_y + cs_y^2, \end{aligned} \tag{3.10}$$

gives (again suppressing lower order terms)

$$Au_{rr} + Bu_{rs} + Cu_{ss} = 0,$$

whose type is given by

$$B^2 - 4AC = (b^2 - 4ac)\left(r_xs_y - r_ys_x\right)^2,$$

from which we deduce that

$$\begin{aligned} b^2 - 4ac &> 0, \quad \Rightarrow \quad B^2 - 4AC > 0, \\ b^2 - 4ac &= 0, \quad \Rightarrow \quad B^2 - 4AC = 0, \\ b^2 - 4ac &< 0, \quad \Rightarrow \quad B^2 - 4AC < 0, \end{aligned}$$

giving that the equation type is unchanged under transformation. We now consider transformations to canonical form. As there are three types of canonical forms, hyperbolic, parabolic and elliptic, we will deal with each type separately.

3.2 Canonical Forms

If we introduce the change of coordinates

$$r = r(x, y), \qquad s = s(x, y), \tag{3.11}$$

the derivatives change according to:
First Order

$$u_x = u_r r_x + u_s s_x, \qquad u_y = u_r r_y + u_s s_y, \tag{3.12}$$

Second Order

$$\begin{aligned} u_{xx} &= u_{rr} r_x^2 + 2u_{rs} r_x s_x + u_{ss} s_x^2 + u_r r_{xx} + u_s s_{xx}, \\ u_{xy} &= u_{rr} r_x r_y + u_{rs}(r_x s_y + r_y s_x) + u_{ss} s_x s_y + u_r r_{xy} + u_s s_{xy}, \\ u_{yy} &= u_{rr} r_y^2 + 2u_{rs} r_y s_y + u_{ss} s_y^2 + u_r r_{yy} + u_s s_{yy}, \end{aligned} \tag{3.13}$$

If we substitute (3.12) and (3.13) into the general linear equation (3.1) and re-arrange we obtain

$$\begin{aligned} (ar_x^2 &+ br_x r_y + cr_y^2) u_{rr} + \left(2ar_x s_x + b(r_x s_y + r_y s_x) + 2cr_y s_y\right) u_{rs} \\ &+ (as_x^2 + bs_x s_y + cs_y^2) u_{ss} + \text{l.o.t.s.} = 0. \end{aligned} \tag{3.14}$$

Our goal now is to target a given canonical form and solve a set of equations for the new coordinates r and s.

3.2.1 Parabolic Canonical Form

Comparing (3.14) with the parabolic canonical form (3.5b) leads to choosing

$$ar_x^2 + br_x r_y + cr_y^2 = 0, \tag{3.15a}$$

$$2ar_x s_x + b(r_x s_y + r_y s_x) + 2cr_y s_y = 0, \tag{3.15b}$$

Since in the parabolic case $b^2 - 4ac = 0$, then substituting $c = \frac{b^2}{4a}$ we find both equations of (3.15) are satisfied if

$$2ar_x + br_y = 0. \tag{3.16}$$

with the choice of $s(x, y)$ arbitrary. The following examples demonstrate.

Example 1.
Consider

$$u_{xx} + 6u_{xy} + 9u_{yy} = 0. \tag{3.17}$$

Here, $a = 1$, $b = 6$ and $c = 9$ showing that $b^2 - 4ac = 0$, so the PDE is parabolic. Solving

$$r_x + 3r_y = 0,$$

gives

$$r = f(3x - y).$$

As we wish to find new coordinates as to transform the original equation to canonical form, we choose

$$r = 3x - y, \quad s = y.$$

Calculating second derivatives

$$u_{xx} = 9u_{rr}, \quad u_{xy} = -3u_{rr} + 3u_{rs}, \quad u_{yy} = u_{rr} - 2u_{rs} + u_{ss}. \tag{3.18}$$

Substituting (3.18) into (3.17) gives

$$u_{ss} = 0!^{\dagger}$$

† Not to be confused with factorial (!).

Solving gives

$$u = f(r)s + g(r).$$

where f and g are arbitrary functions. In terms of the original variables, we obtain the solution

$$u = yf(3x - y) + g(3x - y).$$

Example 2.
Consider

$$x^2 u_{xx} - 4xy u_{xy} + 4y^2 u_{yy} + x u_x = 0. \tag{3.19}$$

Here, $a = x^2$, $b = -4xy$ and $c = 4y^2$ showing that $b^2 - 4ac = 0$, so the PDE is parabolic. Solving

$$x^2 r_x - 2xy r_y = 0,$$

or

$$x r_x - 2y r_y = 0,$$

gives

$$r = f(x^2 y).$$

As we wish to find new coordinates, *i.e.* r and s, we choose simple

$$r = x^2 y, \quad s = y.$$

Calculating first derivatives gives

$$u_x = 2xy u_r. \tag{3.20}$$

Calculating second derivatives

$$u_{xx} = 4x^2 y^2 u_{rr} + 2y u_r, \tag{3.21a}$$

$$u_{xy} = 2x^3 y u_{rr} + 2xy u_{rs} + 2x u_r, \tag{3.21b}$$

$$u_{yy} = x^4 u_{rr} + 2x^2 u_{rs} + u_{ss}. \tag{3.21c}$$

Substituting (3.20) and (3.21) into (3.19) gives

$$4y^2 u_{ss} - 4x^2 y u_r = 0.$$

or, in terms of the new variables, r and s,

$$u_{ss} - \frac{r}{s^2}u_r = 0. \tag{3.22}$$

An interesting question is whether different choices of the arbitrary function f and the variable s would lead to a different canonical forms. For example, suppose we chose

$$r = 2\ln x + \ln y, \qquad s = \ln y,$$

we would obtain

$$u_{ss} - u_r - u_s = 0, \tag{3.23}$$

a constant coefficient parabolic equation, whereas, choosing

$$r = 2\ln x + \ln y, \qquad s = 2\ln x,$$

we would obtain

$$u_{ss} - u_r = 0, \tag{3.24}$$

the heat equation.

3.2.2 Hyperbolic Canonical Form

In order to obtain the canonical form for the hyperbolic type, *i.e.*

$$u_{rr} - u_{ss} + \text{l.o.t.s.} = 0, \tag{3.25}$$

it is necessary to choose

$$\begin{aligned} ar_x^2 + br_xr_y + cr_y^2 &= -\left(as_x^2 + bs_xs_y + cs_y^2\right), \\ 2ar_xs_x + b(r_xs_y + r_ys_x) + 2cr_ys_y &= 0. \end{aligned} \tag{3.26}$$

The problem is that this system is still a very hard problem to solve (both PDEs are nonlinear and coupled!). Therefore, we introduce a modified hyperbolic form that is much easier to work with.

3.2.3 Modified Hyperbolic Form

The modified hyperbolic canonical form is defined as

$$u_{rs} + \text{l.o.t.s.} = 0, \tag{3.27}$$

noting that $a = 0$, $b = 1$ and $c = 0$ and that $b^2 - 4ac > 0$ still! In order to target the modified hyperbolic form, it is now necessary to choose

$$ar_x^2 + br_xr_y + cr_y^2 = 0, \tag{3.28a}$$

$$as_x^2 + bs_xs_y + cs_y^2 = 0. \tag{3.28b}$$

If we re-write (3.28a) and (3.28b) as follows

$$a\left(\frac{r_x}{r_y}\right)^2 + 2b\,\frac{r_x}{r_y} + c = 0, \tag{3.29a}$$

$$a\left(\frac{s_x}{s_y}\right)^2 + 2b\,\frac{s_x}{s_y} + c = 0, \tag{3.29b}$$

then we can solve equations (3.29a) and (3.29b) separately for $\frac{r_x}{r_y}$ and $\frac{s_x}{s_y}$. This leads to two first order linear PDEs for r and s. The solutions of these then gives rise to the correct canonical variables. The following examples demonstrate.

Example 3.
Consider

$$u_{xx} - 5u_{xy} + 6u_{yy} = 0 \tag{3.30}$$

Here, $a = 1$, $b = -5$ and $c = 6$ showing that $b^2 - 4ac = 1 > 0$, so the PDE is hyperbolic. Thus, (3.57a) becomes

$$r_x^2 - 5r_xr_y + 6r_y^2 = 0, \quad s_x^2 - 5s_xr_y + 6s_y^2 = 0,$$

and factoring gives

$$\left(r_x - 2r_y\right)\left(r_x - 3r_y\right) = 0, \quad \left(s_x - 2s_y\right)\left(s_x - 3s_y\right) = 0,$$

from which we choose

$$r_x - 2r_y = 0, \quad s_x - 3s_y = 0,$$

giving rise to solutions

$$r = f(2x + y), \quad s = g(3x + y).$$

As we wish to find new co-ordinates as to transform the original equation to canonical form, we choose

$$r = 2x + y, \quad s = 3x + y.$$

Calculating second derivatives

$$\begin{aligned} u_{xx} &= 4u_{rr} + 12u_{rs} + 9u_{ss}, \\ u_{xy} &= 2u_{rr} + 5u_{rs} + 3u_{ss}, \\ u_{yy} &= u_{rr} + 2u_{rs} + u_{ss}. \end{aligned} \tag{3.31}$$

Substituting (3.31) into (3.30) gives

$$u_{rs} = 0.$$

Solving gives

$$u = f(r) + g(s).$$

where f and g are arbitrary functions. In terms of the original variables, we obtain the solution

$$u = f(2x + y) + g(3x + y).$$

Example 4.

Consider

$$xu_{xx} - (x + y)u_{xy} + yu_{yy} = 0. \tag{3.32}$$

Here, $a = x$, $b = -(x + y)$ and $c = y$ showing that $b^2 - 4ac = (x - y)^2 > 0$, so the PDE is hyperbolic. Solving

$$xr_x^2 - (x + y)r_x r_y + yr_y^2 = 0,$$

or, upon factoring

$$\left(xr_x - yr_y\right)\left(r_x - r_y\right) = 0.$$

As s satisfies the same equation, we choose the first factor for r and the second for s

$$xr_x - yr_y = 0, \quad s_x - s_y = 0. \tag{3.33}$$

Upon solving (3.33), we obtain

$$r = f(xy), \qquad s = g(x+y).$$

As we wish to find new co-ordinates, *i.e.* r and s, we choose simple

$$r = xy, \qquad s = x + y.$$

Calculating first derivatives gives

$$u_x = yu_r + u_s, \qquad u_y = xu_r + u_s. \tag{3.34}$$

Calculating second derivatives

$$u_{xx} = y^2 u_{rr} + 2yu_{rs} + u_{ss}, \tag{3.35a}$$

$$u_{xy} = xyu_{rr} + (x+y)u_{rs} + u_{ss} + u_r, \tag{3.35b}$$

$$u_{yy} = x^2 u_{rr} + 2xu_{rs} + u_{ss}. \tag{3.35c}$$

Substituting (3.34) and (3.35) into (3.32) gives

$$\left(4xy - (x+y)^2\right) u_{rs} - (x+y)u_r = 0,$$

or, in terms of the new variables, r and s,

$$u_{rs} + \frac{s}{s^2 - 4r} u_r = 0.$$

3.2.4 Regular Hyperbolic Form

We now wish to transform a given hyperbolic PDE to its regular canonical form

$$u_{rr} - u_{ss} + \text{l.o.t.s.} = 0. \tag{3.36}$$

First, let us consider the following example.

$$x^2u_{xx} - y^2u_{yy} = 0. \tag{3.37}$$

If we were to transform to modified canonical form, we would solve

$$xr_x - yr_y = 0, \quad xs_x + ys_y = 0,$$

which gives

$$r = f(xy), \qquad s = g(x/y).$$

As we wish to find new co-ordinates, *i.e.* r and s, we choose simple

$$r = xy, \qquad s = x/y.$$

In doing so, the original PDE then becomes

$$u_{rs} - \frac{1}{2r}u_s = 0. \tag{3.38}$$

However, if we choose

$$r = \ln x + \ln y, \qquad s = \ln x - \ln y,$$

then the original PDE becomes

$$u_{rs} - u_s = 0, \tag{3.39}$$

which is clearly an easier PDE. However, if we introduce new coordinates α and β such that

$$\alpha = \frac{r+s}{2}, \qquad \beta = \frac{r-s}{2},$$

noting that derivatives transform

$$u_r = \frac{1}{2}u_\alpha + \frac{1}{2}u_\beta, \quad u_s = \frac{1}{2}u_\alpha - \frac{1}{2}u_\beta \quad u_{rs} = \frac{1}{4}u_{\alpha\alpha} - \frac{1}{4}u_{\beta\beta}, \tag{3.40}$$

and the PDE (3.39) becomes

$$u_{\alpha\alpha} - u_{\beta\beta} - 2u_\alpha + 2u_\beta = 0,$$

a PDE in regular hyperbolic form. Thus, combining the variables r and s and α and β gives directly

$$\alpha = \ln x, \quad \beta = \ln y.$$

In fact, one can show that if

$$\alpha = \frac{r+s}{2}, \quad \beta = \frac{r-s}{2},$$

where r and s satisfies (3.28a) and (3.28b) then α and β satisfies

$$a\alpha_x^2 + b\alpha_x\alpha_y + c\alpha_y^2 = -\left(a\beta_x^2 + b\beta_x\beta_y + c\beta_y^2\right), \tag{3.41a}$$

$$2a\alpha_x\beta_x + b(\alpha_x\beta_y + \alpha_y\beta_x) + 2c\alpha_y\beta_y = 0. \tag{3.41b}$$

which is (3.58a) with r and s replaces with α and β. This give a convenient way to go directly to the coordinates that lead to the regular hyperbolic form. We note that

$$\alpha, \beta = \frac{r \pm s}{2}, \tag{3.42}$$

so we can essentially consider

$$ar_x^2 + br_xr_y + cr_y^2 = 0, \tag{3.43a}$$

$$as_x^2 + bs_xs_y + cs_y^2 = 0. \tag{3.43b}$$

but instead of factoring, treat each as a quadratic equation in r_x/r_y or s_x/s_y and solve according. We demonstrate with an example.

Example 5.

Consider

$$8u_{xx} - 6u_{xy} + u_{yy} = 0. \tag{3.44}$$

The corresponding equations for r and s are

$$8r_x^2 - 6r_xr_y + r_y^2 = 0, \tag{3.45a}$$

$$8s_x^2 - 6s_xs_y + s_y^2 = 0, \tag{3.45b}$$

but as they are identical it suffices to only consider one. Dividing (3.57a) by r_y^2 gives

$$8\left(\frac{r_x}{r_y}\right)^2 - 6\frac{r_x}{r_y} + 1 = 0.$$

Solving by the quadratic formula gives

$$\frac{r_x}{r_y} = \frac{6 \pm 2}{16},$$

or

$$8r_x - (3 \pm 1)r_y = 0.$$

The method of characteristics gives

$$\frac{dx}{8} = -\frac{dy}{3 \pm 1}; \quad dr = 0.$$

which gives

$$r = f\left((3 \pm 1)x + 8y\right),$$

which we choose

$$r = 3x + 8y \pm x,$$

which leads to the chose

$$r = 3x + 8y, \qquad s = x,$$

Under this transformation, the original equation (3.44) becomes

$$u_{rr} - u_{ss} = 0,$$

the desired canonical form.

Example 6.

Consider

$$xy^3u_{xx} - x^2y^2u_{xy} - 2x^3yu_{yy} - y^2u_x + 2x^2u_y = 0. \tag{3.46}$$

The corresponding equations for r and s are

$$xy^3r_x^2 - x^2y^2r_xr_y - 2x^3yr_y^2 = 0, \tag{3.47a}$$

$$xy^3s_x^2 - x^2y^2s_xr_y - 2x^3ys_y^2 = 0, \tag{3.47b}$$

and choosing the first gives

$$y^2\left(\frac{r_x}{r_y}\right)^2 - xy\frac{r_x}{r_y} - 2x^2 = 0.$$

Solving by the quadratic formula gives

$$\frac{r_x}{r_y} = \frac{(1\pm 3)x}{2y},$$

or

$$2yr_x - (1\pm 3)\,xr_y = 0.$$

Solving gives

$$r = f\left(x^2 + 2y^2 \pm 3x^2\right).$$

If we choose f to be simple and split according to the $\pm$ gives

$$r = x^2 + 2y^2, \quad s = 3x^2,$$

Under this transformation, the original equation (3.46) becomes

$$u_{rr} - u_{ss} = 0,$$

the desired canonical form.

3.2.5 Elliptic Canonical Form

In order to obtain the canonical form for the elliptic type, *i.e.*

$$u_{rr} + u_{ss} + \text{l.o.t.s.} = 0,$$

it is necessary to choose

$$\begin{aligned} ar_x^2 + br_xr_y + cr_y^2 &= \left(as_x^2 + bs_xs_y + cs_y^2\right), \\ 2ar_xs_x + b(r_xs_y + r_ys_x) + 2cr_ys_y &= 0. \end{aligned} \tag{3.48}$$

The problem, like the regular hyperbolic type, is still difficult to solve. However, we find that if we let †

$$r = \frac{\alpha+\beta}{2}, \quad s = \frac{\alpha-\beta}{2i} \tag{3.49}$$

†Please note the switch in the variables r and s and α and β.

where α and β satisfy

$$a\alpha_x^2 + b\alpha_x\alpha_y + c\alpha_y^2 = 0, \tag{3.50a}$$

$$a\beta_x^2 + b\beta_x\beta_y + c\beta_y^2 = 0, \tag{3.50b}$$

then (3.48) is satisfied. This is much like the connection between modified and regular hyperbolic canonical form. As solving (3.50a) gives rise to complex roots, the formulas (3.49) will take real and complex parts of the solved $\alpha and \beta$ equations as new variables. The next few examples will illustrate.

Example 7.
Consider

$$u_{xx} - 4u_{xy} + 5u_{yy} = 0. \tag{3.51}$$

The corresponding equations for r and s are

$$r_x^2 - 4r_xr_y + 5r_y^2 = 0, \tag{3.52a}$$

$$s_x^2 - 4s_xs_y + 5s_y^2 = 0, \tag{3.52b}$$

but as they are identical it suffices to only consider one. Dividing (3.52a) by r_y^2 gives

$$\left(\frac{r_x}{r_y}\right)^2 - 4\frac{r_x}{r_y} + 5 = 0.$$

Solving by the quadratic formula gives

$$\frac{r_x}{r_y} = 2 \pm i,$$

or

$$r_x - (2 \pm i)r_y = 0.$$

The method of characteristics gives

$$\frac{dx}{1} = -\frac{dy}{2 \pm i}, \quad dr = 0.$$

which gives

$$r = f\left(2x + y \pm ix\right),$$

which we choose

$$r = 2x + y \pm x,$$

which leads to the chose

$$r = 2x + y, \qquad s = x,$$

Under this transformation, the original equation (3.51) becomes

$$u_{rr} + u_{ss} = 0,$$

the desired canonical form.

Example 8.

Consider

$$2\left(1+x^2\right)^2 u_{xx} - 2\left(1+x^2\right)\left(1+y^2\right) u_{xy} + \left(1+y^2\right)^2 u_{yy} + 4x\left(1+x^2\right) u_x = 0. \tag{3.53}$$

The corresponding equations for r and s are

$$2\left(1+x^2\right)^2 r_x^2 - 2\left(1+x^2\right)\left(1+y^2\right) r_x r_y + \left(1+y^2\right)^2 r_y^2 = 0, \tag{3.54a}$$

$$2\left(1+x^2\right)^2 s_x^2 - 2\left(1+x^2\right)\left(1+y^2\right) s_x s_y + \left(1+y^2\right)^2 s_y^2 = 0, \tag{3.54b}$$

but as they are identical it suffices to only consider one. Solving by the quadratic formula gives

$$\frac{r_x}{r_y} = \frac{(2 \pm i)\left(1+y^2\right)}{1+x^2},$$

or

$$2\left(1+x^2\right) r_x - (1 \pm i)\left(1+y^2\right) r_y = 0.$$

The method of characteristics gives the solution as

$$r = f\left(\tan^{-1} x + 2\tan^{-1} y \pm \tan^{-1} x\right),$$

which we choose

$$r = \tan^{-1} x + 2\tan^{-1} y \pm \tan^{-1} x,$$

which leads to the choice

$$r = \tan^{-1} x + 2\tan^{-1} y, \quad s = \tan^{-1} x,$$

Under this transformation, the original equation (3.53) becomes

$$u_{rr} + u_{ss} - 2yu_r = 0,$$

and upon using the original transformation gives

$$u_{rr} + u_{ss} - 2\tan\frac{r-s}{2}u_r = 0,$$

the desired canonical form.

Exercises

1. Determine the type of the following second order PDEs

(i) $x^2 u_{xx} - y^2 u_{yy} = u_x + u_y$
(ii) $u_{xx} + 2u_{xy} + u_{yy} = 0$
(iii) $y^2 u_{xx} + 2yu_{xy} - u_{yy} = 0$
(iv) $u_{xy} + u = u_x + u_y$

2. Transform the following parabolic PDEs to canonical form. Find the general solution if possible.

(i) $u_{xx} + 2u_{xy} + u_{yy} = 0$
(ii) $y^2 u_{xx} - 2xyu_{xy} + x^2 u_{yy} = 0.$
(iii) $y^2 u_{xx} + 2xyu_{xy} + x^2 u_{yy} - 2xu_x = 0.$

3. Reduce the following second order PDEs to modified hyperbolic canonical form

(i) $2u_{xx} - 3u_{xy} + u_{yy} = u_x + u_y,$
(ii) $x^2 u_{xx} - 3xyu_{xy} + 2y^2 u_{yy} = 0.$

4. Reduce the following second order PDEs to canonical form

(i) $4u_{xx} - 8u_{xy} + 3u_{yy} = 0$,
(ii) $4u_{xx} + 4u_{xy} + 5u_{yy} = 1$,
(iii) $x^2 u_{xx} + y^2 u_{yy} = 1$,
(iv) $u_{xx} - (1+y^2)^2 u_{yy} = 0$.

5. If a linear second order PDE

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + \text{lots} = 0.$$

is hyperbolic, then it is possible to transform to a modified hyperbolic canonical form

$$u_{rs} + \text{l.o.t.s.} = 0,$$

by choosing r and s such that they satisfy

$$\begin{aligned} ar_x^2 + 2br_x r_y + cr_y^2 &= 0, \\ as_x^2 + 2bs_x s_y + cs_y^2 &= 0. \end{aligned}$$

5(i) Show that by introducing new variables α and β such that

$$\alpha = r + s, \quad \beta = r - s,$$

then the following equations are satisfied

$$\begin{aligned} a\alpha_x^2 + 2b\alpha_x\alpha_y + c\alpha_y^2 &= -\left(a\beta_x^2 + 2b\beta_x\beta_y + c\beta_y^2\right), \\ a\alpha_x\beta_x + b(\alpha_x\beta_y + r_y\beta_x) + c\alpha_y\beta_y &= 0. \end{aligned}$$

This then leads to the hyperbolic canonical form

$$u_{\alpha\alpha} - u_{\beta\beta} + \text{l.o.t.s.} = 0.$$

5(ii). If, instead of the α and β introduced in 5(i), we introduce α and β such that

$$\alpha = c_{11}r + c_{12}s, \quad \beta = c_{21}r + c_{22}s,$$

where c_{11}, c_{12}, c_{21} and c_{22} are constant and r and s satisfy the above first order PDEs, namely

$$\begin{aligned} ar_x^2 + 2br_xr_y + cr_y^2 &= 0, \\ as_x^2 + 2bs_xs_y + cs_y^2 &= 0, \end{aligned}$$

find conditions on c_{11}, c_{12}, c_{21} and c_{22} such that

$$\begin{aligned} a\alpha_x^2 + 2b\alpha_x\alpha_y + c\alpha_y^2 &= -\left(a\beta_x^2 + 2b\beta_x\beta_y + c\beta_y^2\right), \\ a\alpha_x\beta_x + b(\alpha_x\beta_y + r_y\beta_x) + c\alpha_y\beta_y &= 0. \end{aligned}$$

are still satisfied.