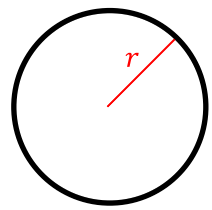
Name ___Solutions____ Calculus I, Test 2, Spring 2017

Please show all your work and circle your answer when appropriate. You do not need to simplify answers unless the problem specifies to do so.

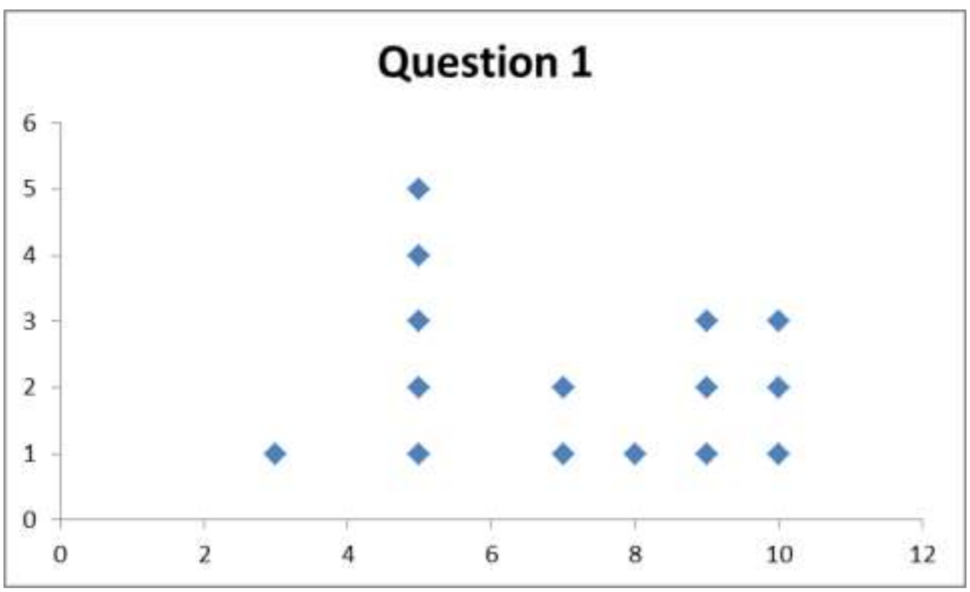
1) A circle has an initial radius of 50 feet when the radius begins to decrease at a rate of 2 feet per minute. What is the range of change of the area at the instant the radius is 10 feet? (10 points)

$\frac{dr}{dt} = -2$
$r = 10$
$A = \pi r^2$
$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$



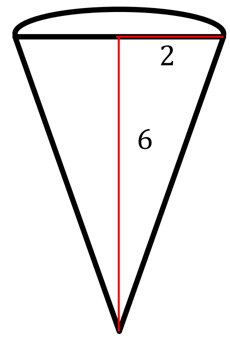
$$\frac{dA}{dt} = 2\pi \cdot 10 \cdot (-2) = -40\pi$$

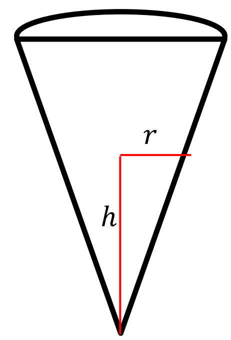
$$-40\pi \text{ft}^2/\text{s}$$



2) A cone-shaped funnel has a height of 6 inches tall and base 4 inches across. That's a diameter, not a radius. It is draining such that the depth decreases a constant rate of $\frac{1}{4}$ in/s. At what rate is the liquid coming out of the bottom when the height of the liquid is $\sqrt{3}$ inches? Give your answer as volume per time. (10 points)

(The volume of a cone is $V = \frac{1}{3}\pi r^2 h$)



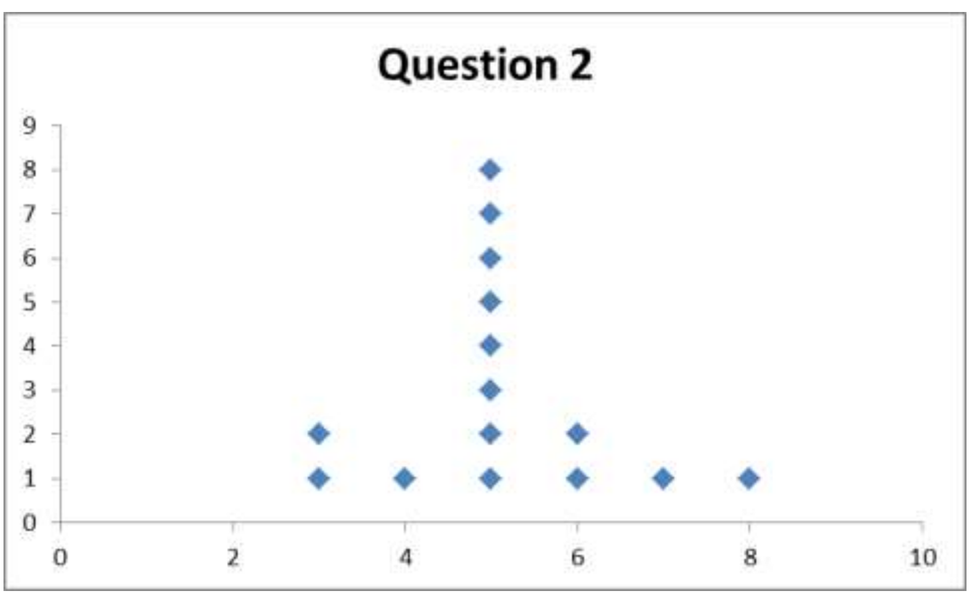


$$\frac{6}{h} = \frac{2}{r}$$
$$6r = 2h$$
$$r = \frac{h}{3}$$

$$V = \frac{1}{3}\pi r^2 h$$
$$h = \sqrt{3}$$
$$\frac{dh}{dt} = -\frac{1}{4}$$

$$V = \frac{1}{3}\pi r^2 h$$
$$V = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{\pi h^3}{27}$$
$$\frac{dV}{dt} = \frac{\pi}{27}\cdot 3h^2\frac{dh}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{9}\left(\sqrt{3}\right)^2\left(-\frac{1}{4}\right) = -\frac{\pi}{12}\ \text{in}^3/\text{s}$$



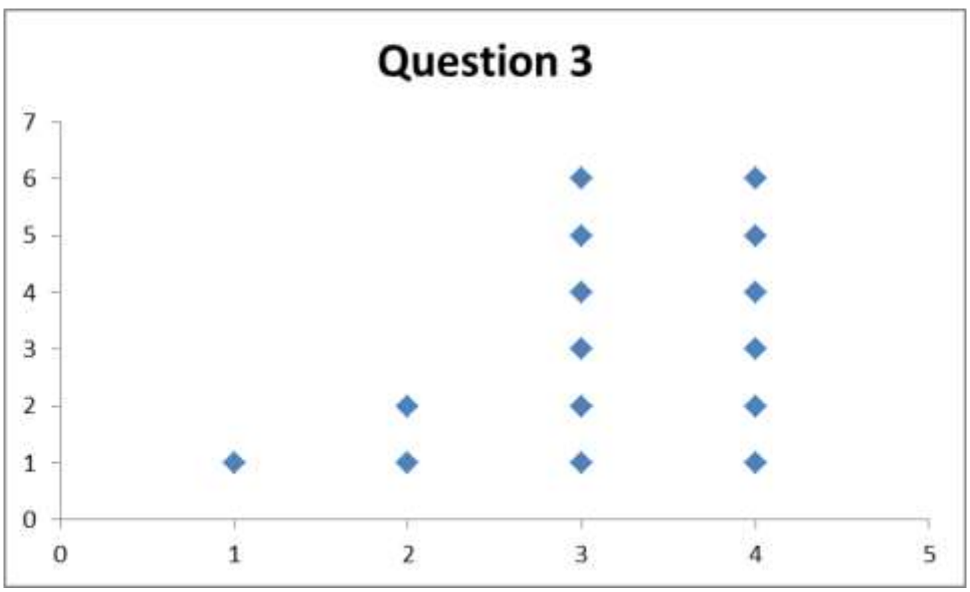
Use the function $f(x) = \frac{4x^5}{5} - 3x^3 + 5$ for the next **TWO** pages. Also note that not every number on this page will be an integer. It'll work out fairly nicely, but will involve square roots

Note that throughout this series of problems, your work was followed. The answer to one question was used to determine the correctness of the next.

3) Find x-values that could possibly be a minimum or maximum (critical values) of the function. (4 points)

CVs occur when the derivative is zero, doesn't exist, or at endpoints of the domain. Here we only have when $f'(x) = 0$.

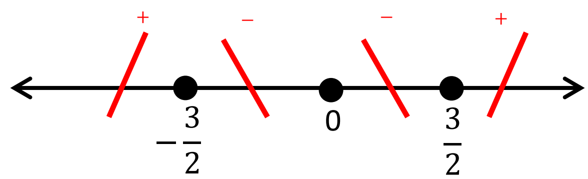
$$f'(x) = 4x^4 - 9x^2$$
$$4x^4 - 9x^2 = 0$$
$$x^2(4x^2 - 9) = 0$$
$$x^2(2x-3)(2x+3) = 0$$
$$x = 0, x = \frac{3}{2}, x = -\frac{3}{2}$$

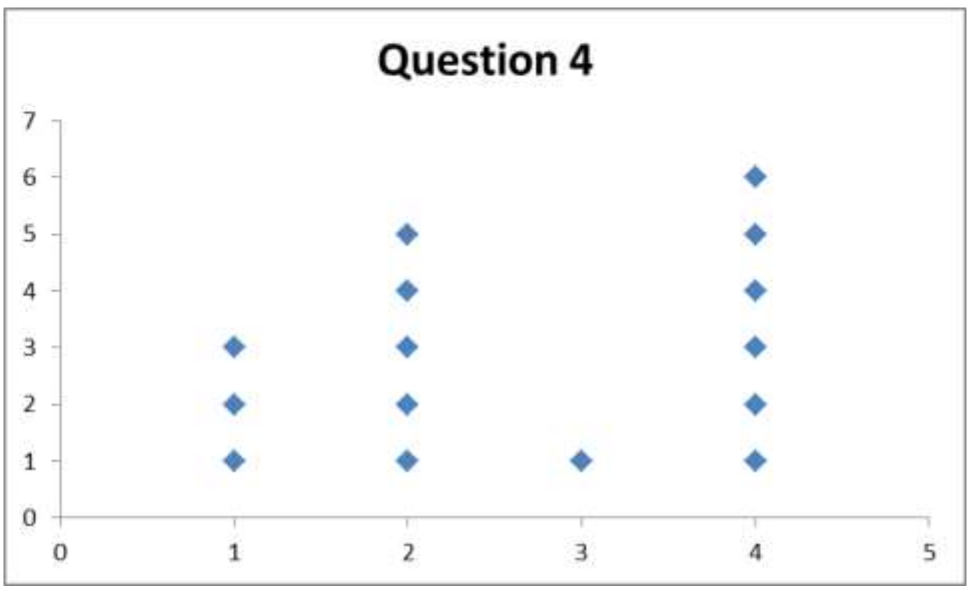


4) For each critical value, determine whether it is a maximizer, minimizer, or neither. (This question has multiple answers. One for each CV) (4 points)

We use the derivative $f'(x) = x^2(2x-3)(2x+3)$ in order determine the signs below. A Positive derivative means the function is increasing. A negative derivative means the function is decreasing.

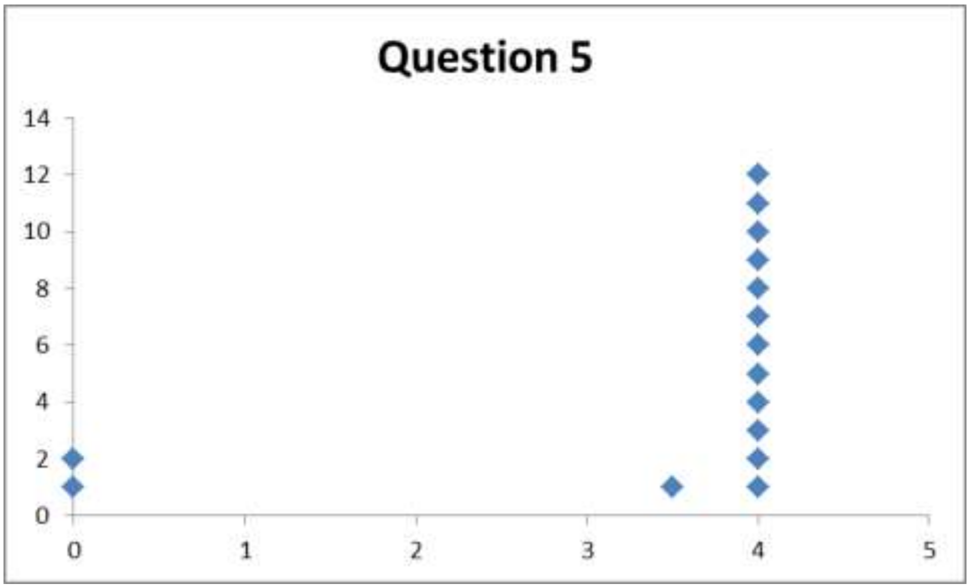
We see from the sign chart that $-\frac{3}{2}$ is a maximum while $\frac{3}{2}$ is a minimum. Curiously, $x = 0$ is neither a maximum nor a minimum. The derivative is zero there, though, so it is a flatspot. We'll need that information later when we graph $y = f(x)$





5) At least one of your CVs should have turned out to be a maximizer. Find the local maximum corresponding to each maximizer. Do not simplify your answer. (4 points)

$x = -\frac{3}{2}$ is a maximizer. The corresponding maximum is the y-value: $f\left(-\frac{3}{2}\right) = \frac{4}{5}\left(-\frac{3}{2}\right)^5 - 3\left(-\frac{3}{2}\right)^3 + 5$

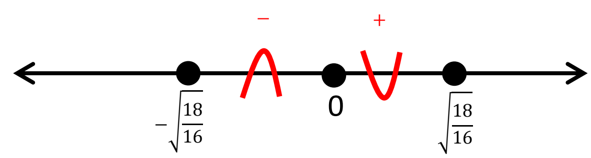


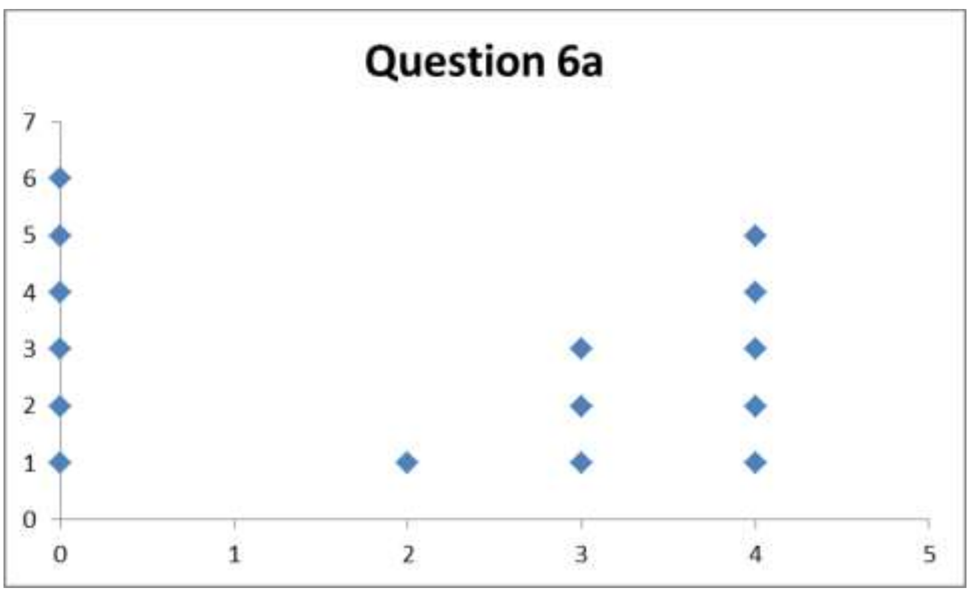
6) Find the x-value for one of the inflection points. Use a sign chart to justify why it is an inflection point. (4 points)

We need the second derivative here:

$$f''(x) = 16x^3 - 18x = x(16x^2 - 18)$$

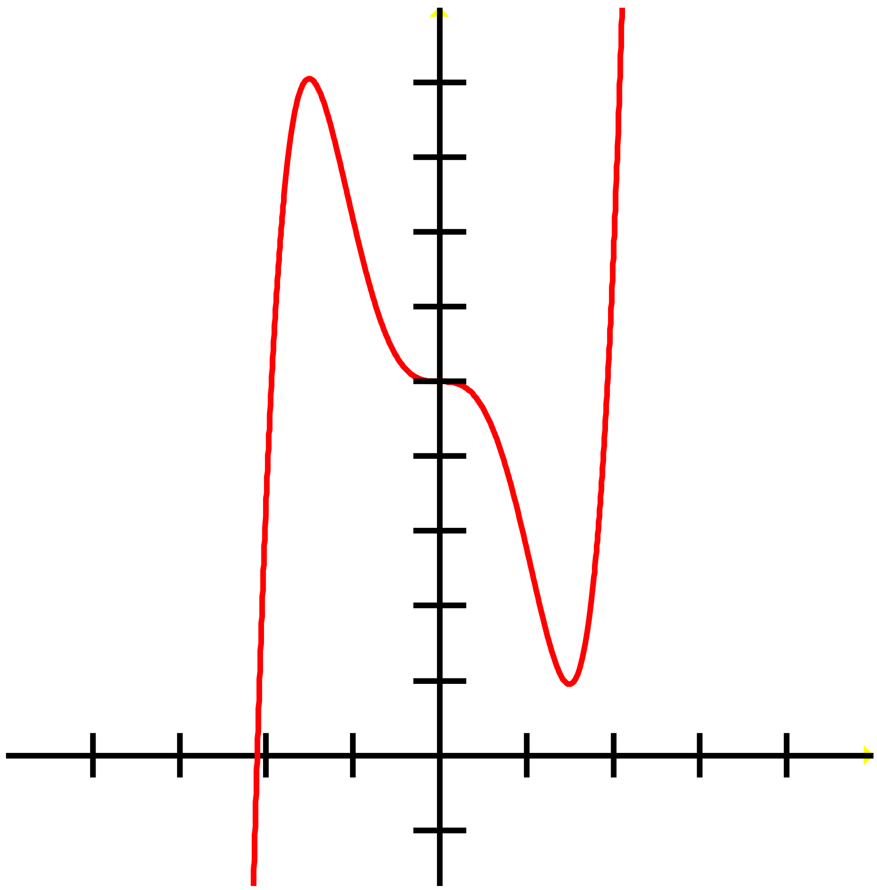
This gives us 3 possible inflection points: $x = 0, \pm\sqrt{\frac{18}{16}}$. Zero is the simplest, so let's choose that one and create a sign chart for the second derivative:

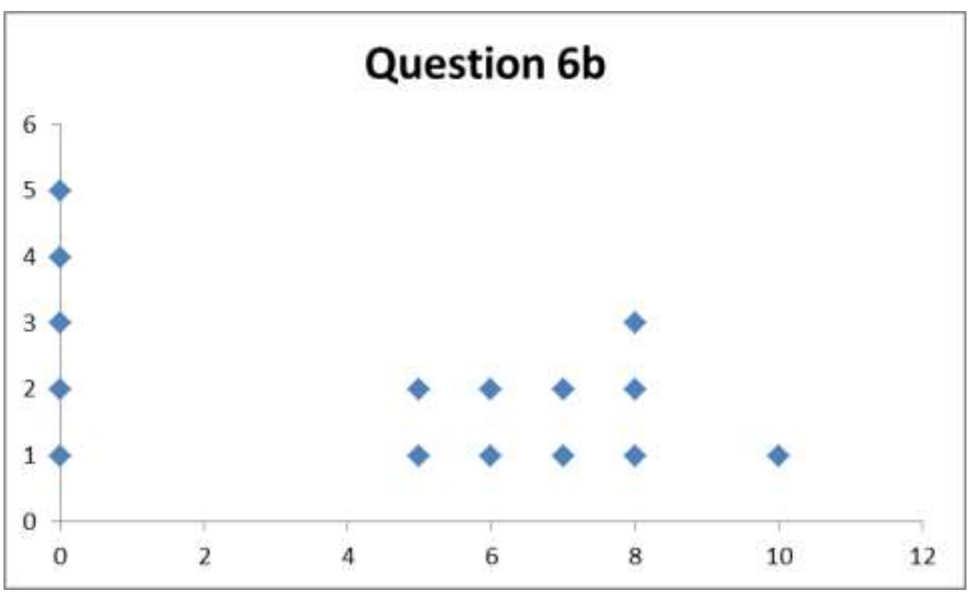




Graph the function. (10 points)

(Accuracy is important. I'll be looking to see that it aligns with all the information above)

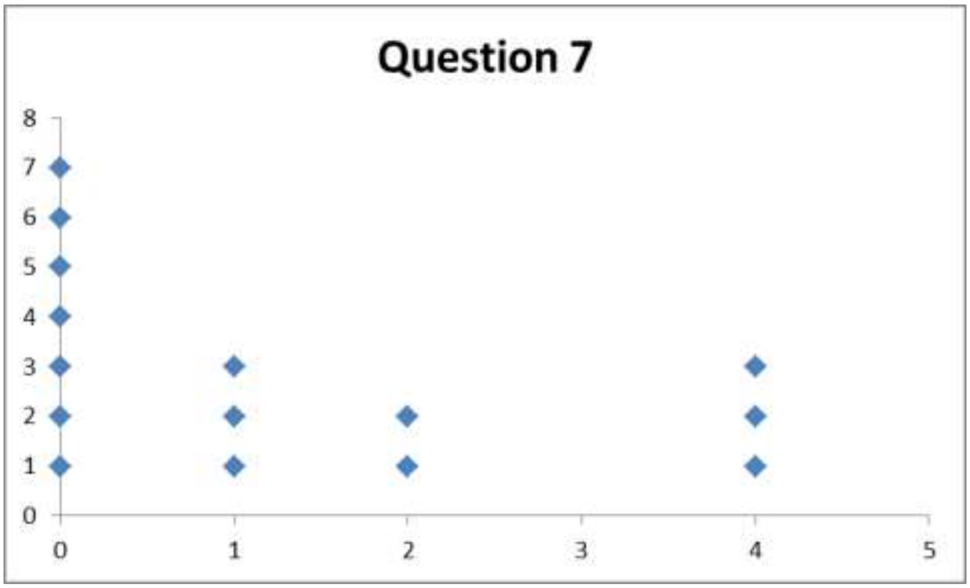




On this page we're going to use the same function: $f(x) = \frac{4x^5}{5} - 3x^3 + 5$. However, now we're going to restrict the domain to $[-2, 3]$

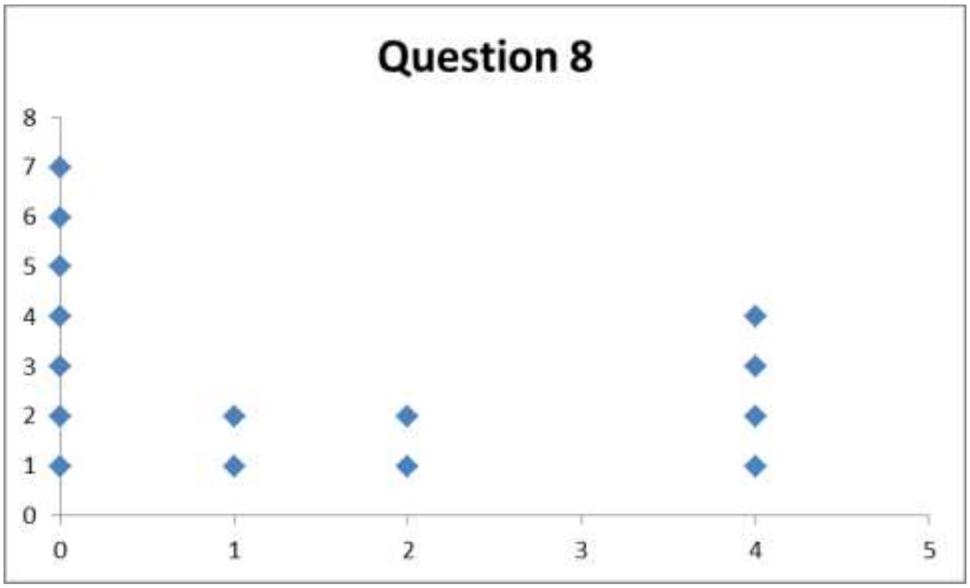
7) Find all local maximizers. (4 points)

The maximizer of $x = -\frac{3}{2}$ is still a maximizer. Now also the endpoint at $x = 3$ is.



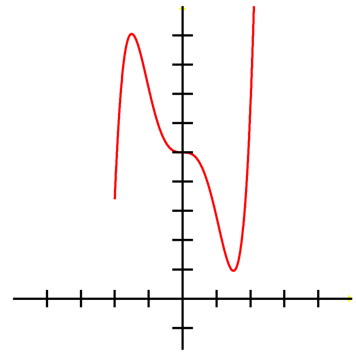
8) Find all local minimizers. (4 points)

The minimizer of $x = \frac{3}{2}$ is still a minimizer. Now also the endpoint at $x = -2$ is.

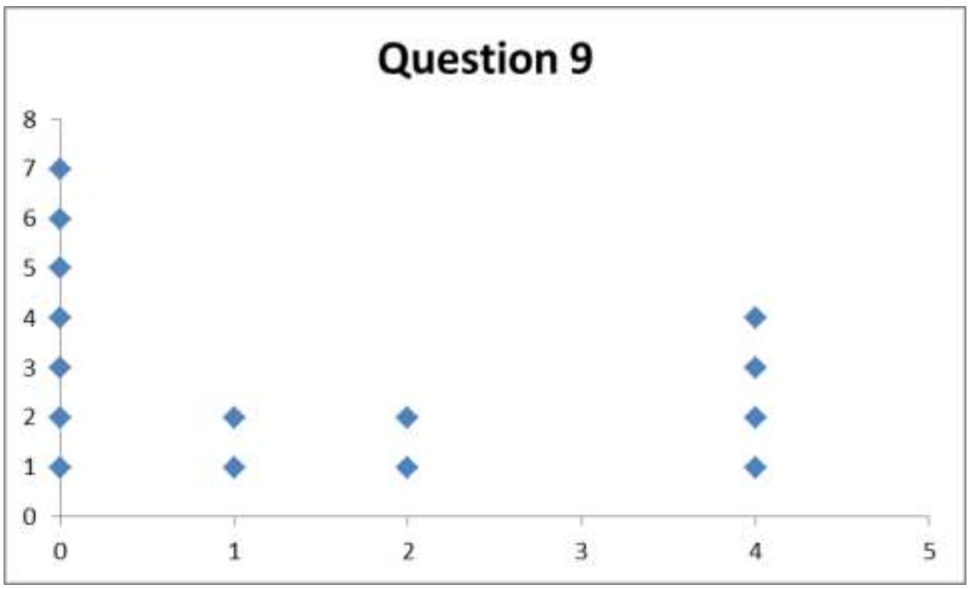


9) Based on the graph you drew, where is the absolute maximum? (If you try to do this algebraically, it will be tedious without a calculator. That's why I'm saying to go back to your graph, and answer this question based on however you drew your function) (4 points)

Let's cut it off at -2 and 3 to see (right)

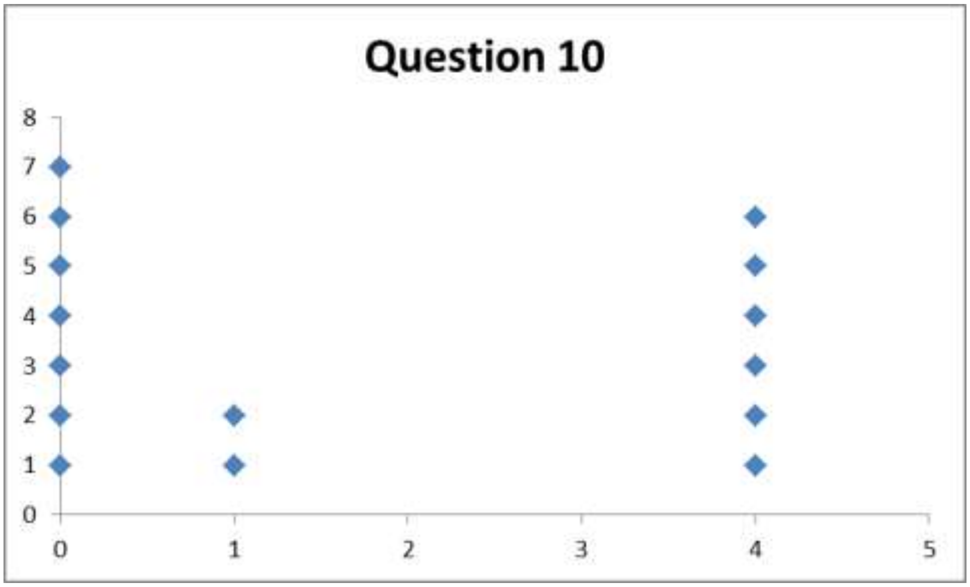


The absolute maximum is at $x = 3$.



10) Based on the graph you drew, where is the absolute minimum? (4 points)

From the graph we see that it is at $x = -2$.



11) Find numbers x and y Satisfying the equation $3x + y = 12$ such that the product of x and y is as large as possible. (8 points)

We want to maximize the expression xy. Let's call it $A = xy$. We want to express this as a function of just one variable, so we use the other equation to help us with that:

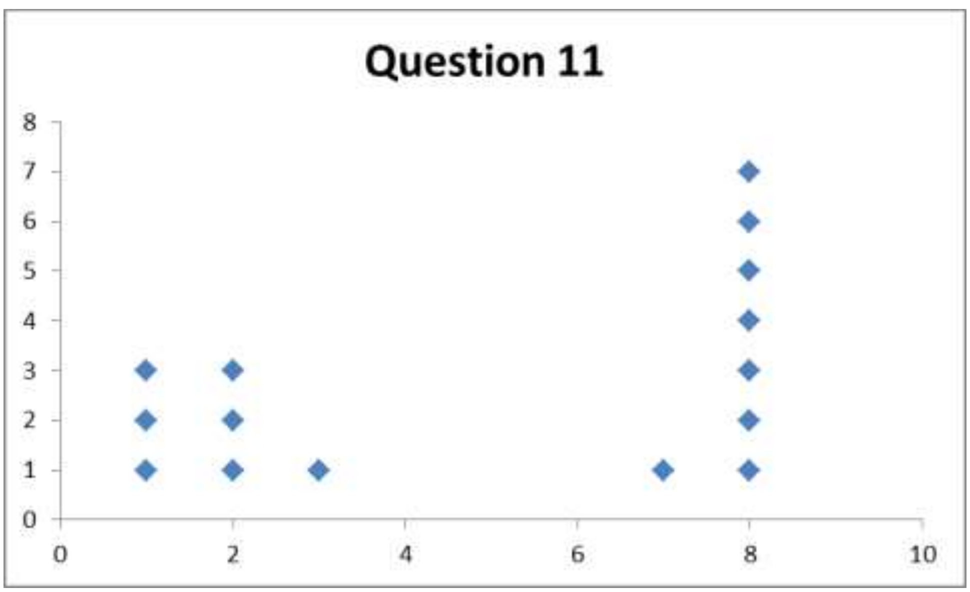
$$3x + y = 12$$
$$y = 12 - 3x$$

$$A = x(12 - 3x) = -3x^2 + 12x$$

$$A' = -6x + 12$$
$$-6x + 12 = 0$$
$$x = 2$$

$$y = 12 - 3x = 12 - 6 = 6$$

The solution is $x = 2; y = 6$.



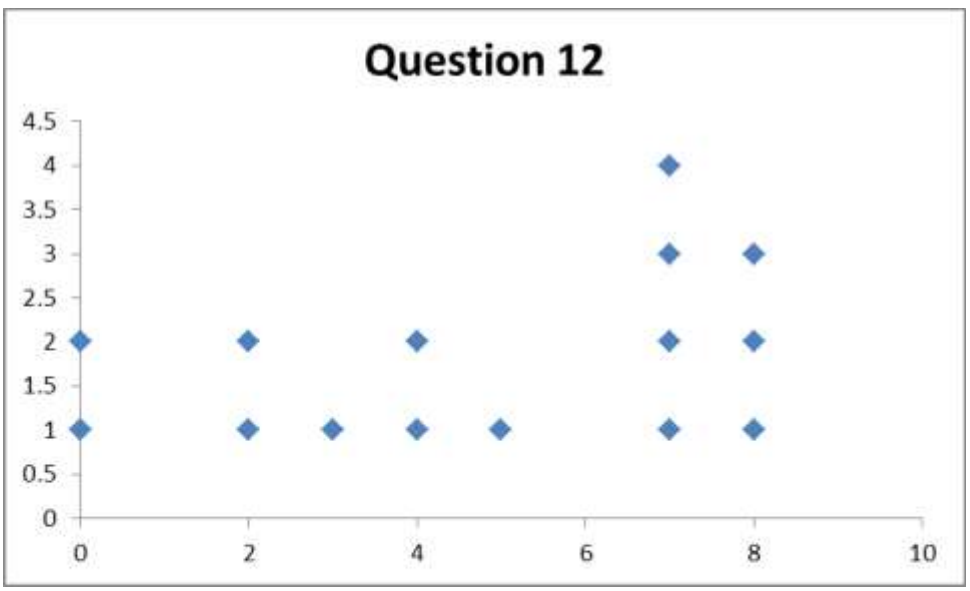
12) An entrepreneur rents batteries at Central Park to Pokémon Go players and has a profit function as given below. How many batteries should they rent to maximize their profit? The profit P is measured in dollars, while the variable b is measured in hundreds of batteries. $0 \leq b \leq 40$. (8 points)

$$P(b) = 32b - b^2$$

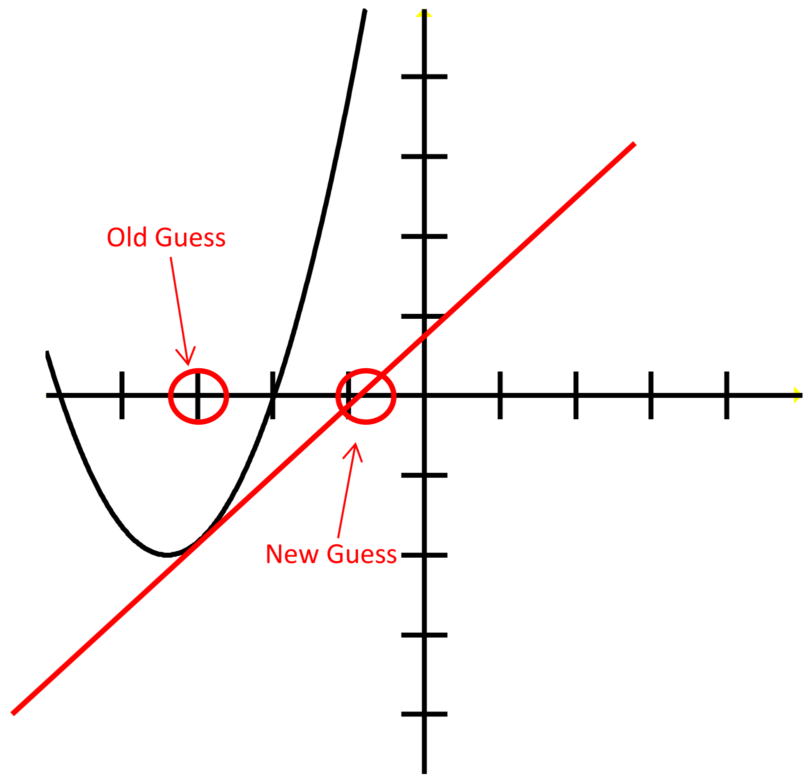
We want to maximize the function $P(b) = 32b - b^2$. We do that by finding the CVs using the derivative:

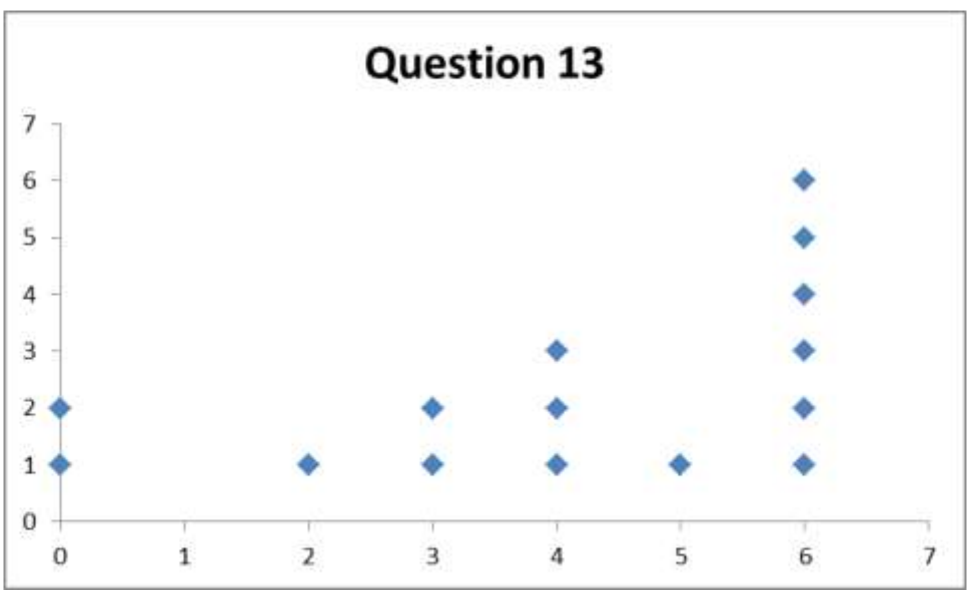
$$P'(b) = 32 - 2b$$
$$32 - 2b = 0$$
$$2b = 32$$
$$b = 16$$

The maximum profit occurs when they rent 1600 batteries.



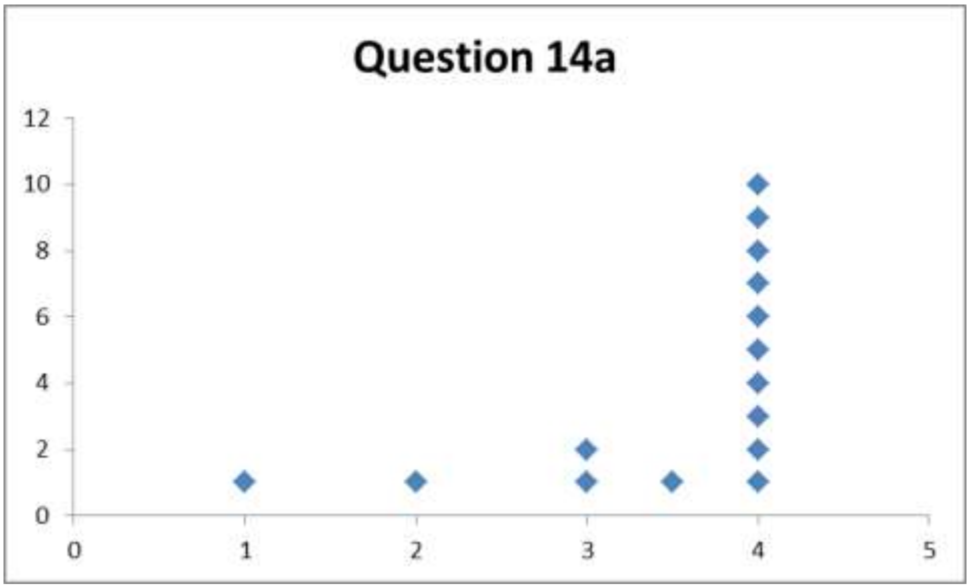
13) Below is a graph of a function. If Dr. Beyerl guesses that a root of the function is $x = -3$, use Newton's Method to improve his guess. Illustrate what you do on the graph, and circle the new guess. (6 points)



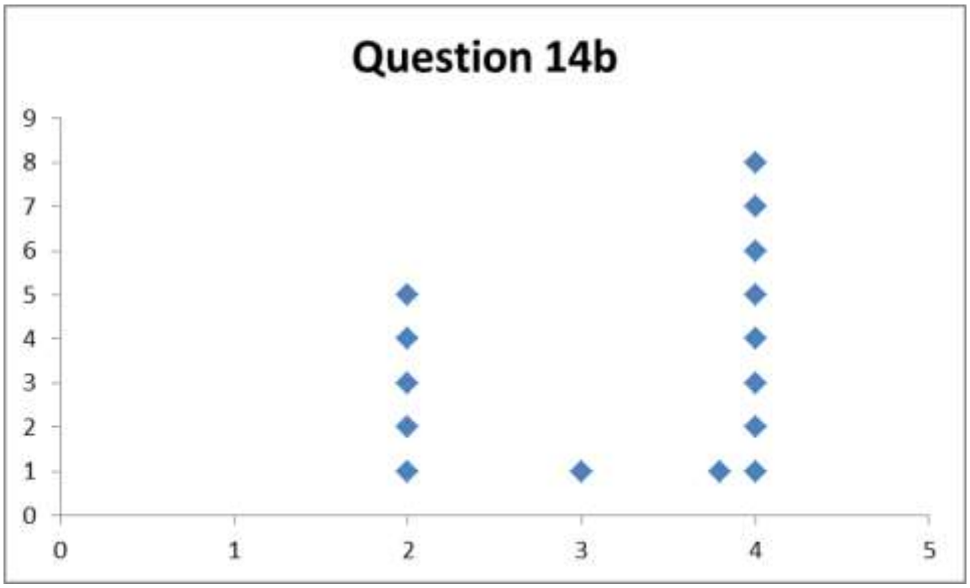


14) Compute each of the limits below. (4 points each)

$$\lim_{x\to 3}\frac{x^{15}-3^{15}}{x^2-9} = \lim_{x\to 3}\frac{15x^{14}}{2x} = \frac{15\cdot 3^{14}}{2\cdot 3}$$

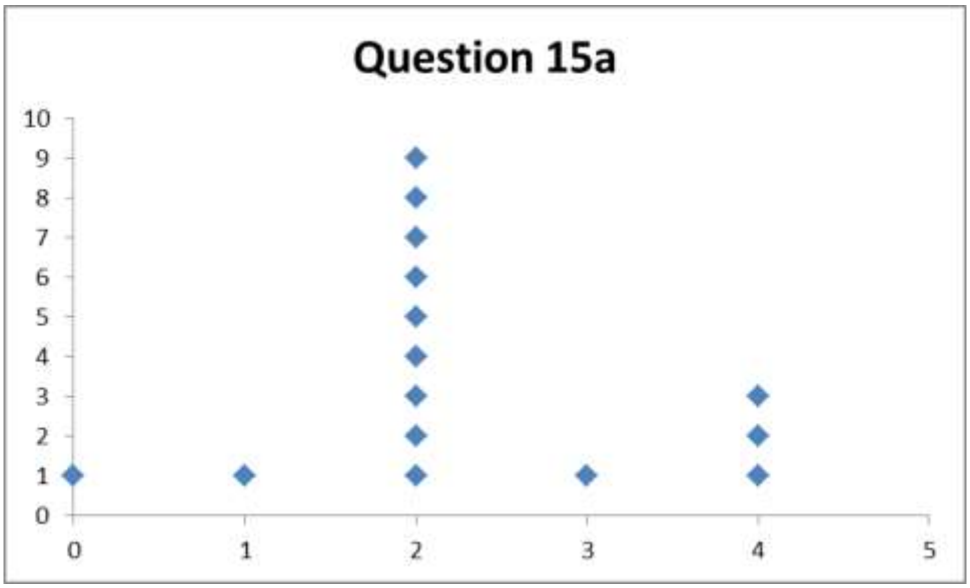


$$\lim_{x\to 0}\frac{\cos(x)-1}{x} = \lim_{x\to 0}\frac{-\sin(x)}{1} = -\sin(0) = 0$$



15) Compute each of the limits below. (4 points each)

$$\lim_{x\to 7}\frac{\ln(x)-\ln(7)}{x-7} = \lim_{x\to 7}\frac{\frac{1}{x}}{1} = \frac{1}{7}$$



$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -\frac{x^2}{x} = \lim_{x\to 0^+} -x = 0$$