Name ____ Test 2, Spring 2022

Part 1: Computational Skills

1) Given the function below, find all minimizers. (6 points)

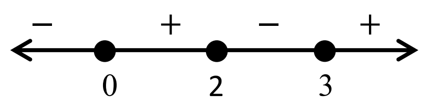
$$f(x) = 3x^4 - 20x^3 + 36x^2 + 4$$

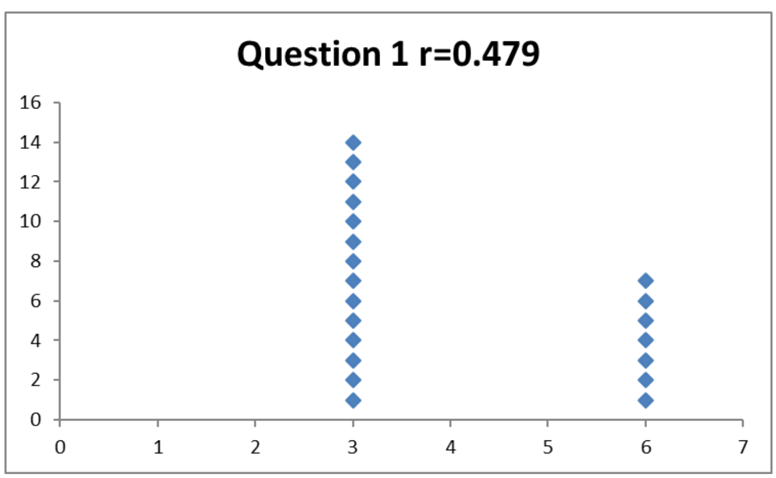
Answer: $\boxed{0, 3}$

$$f'(x) = 12x^3 - 60x^2 + 72x = 12x(x^2 - 5x + 6) = 12x(x-2)(x-3)$$

CVs: $x = 0,2,3$

Minimizers at: $x = 0, 3$





Grading Note: A lot of people didn't notice $x = 0$ is a CV.

2) Given the function below, find all maximizers. (6 points)

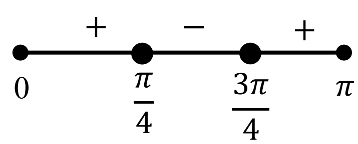
Hint: $\cos^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta)$

$$f(x) = \sin(x)\cos(x)$$
$$x \in [0, \pi]$$

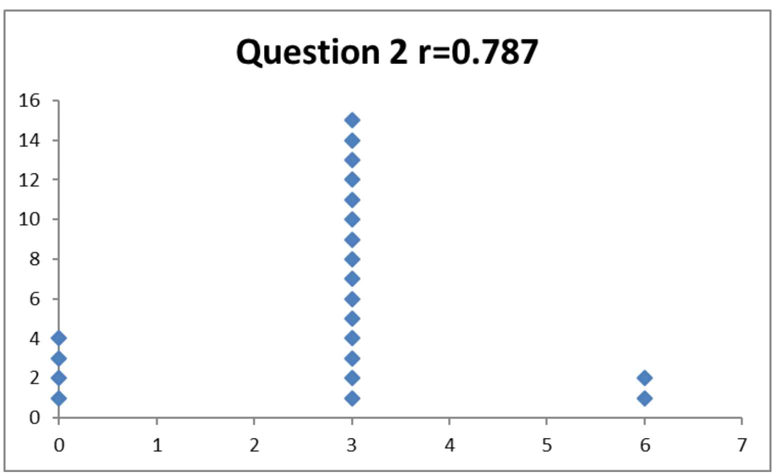
Answer: $\boxed{45^o, 180^o}$

$$f'(x) = \cos(x)\cos(x) + \sin(x)(-\sin(x)) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x)$$

$$1 - 2\sin^2(x) = 0$$
$$2\sin^2(x) = 1$$
$$\sin^2(x) = \frac{1}{2}$$
$$\sin(x) = \frac{1}{\sqrt{2}}$$
$$x = 45^o, 135^o$$



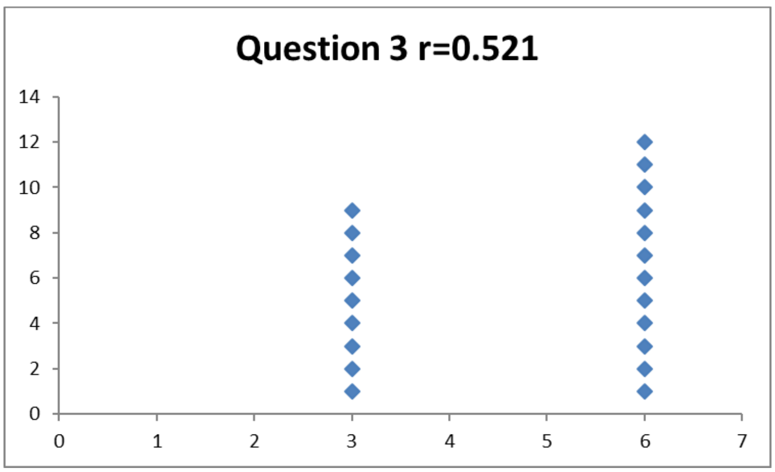
It's clear from the sign chart that $\frac{\pi}{4}$ is a maximum. Don't forget the endpoints: π is as well.



3) Find the general antiderivative of $f(x) = 3x^4 - \frac{2}{x^3}$. (6 points)

Answer: $\boxed{\frac{3}{5}x^5 + \frac{1}{x^2} + C}$

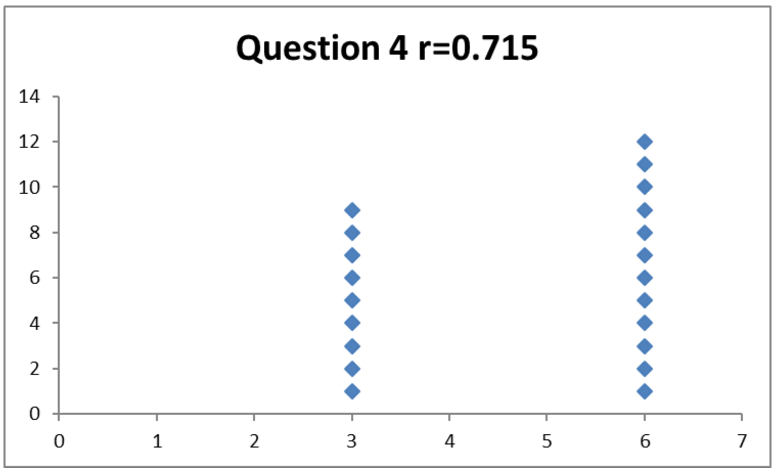
$$f(x) = 3x^4 - 2x^{-3}$$
$$F(x) = \frac{3x^5}{5} - \frac{2x^{-2}}{-2} + C = \frac{3}{5}x^5 + \frac{1}{x^2} + C$$



4) Find the antiderivative of $f(x) = e^{2x}$ such that $F(0) = 4$. (6 points)

Answer: $\boxed{\frac{e^{2x}}{2} + 3.5}$

$$F(x) = e^{2x} \cdot \frac{1}{2} + C = \frac{e^{2x}}{2} + C$$
$$4 = \frac{e^0}{2} + C = \frac{1}{2} + C$$
$$C = 3.5$$



5) A magic cube is growing at a rate of $3\text{in}^3/\text{s}$. When the cube is $4\text{in} \times 4\text{in} \times 4\text{in}$, how quickly are the side lengths increasing? (6 points)

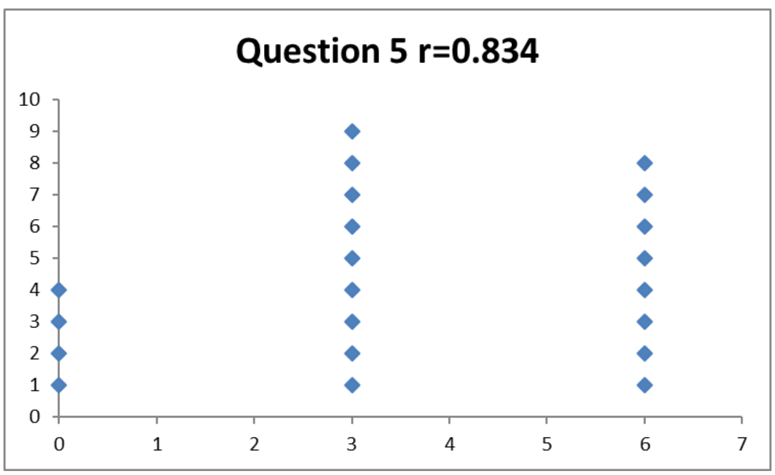
Answer: $\boxed{\frac{1}{16}\text{in/s}}$

Equation
$V = l^3$

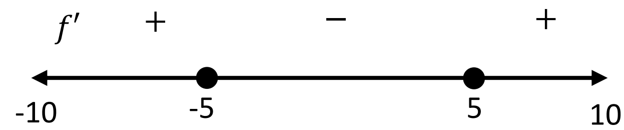
Variables
$V = ??$
$V' = 3$
$l = 4$
$l' = ??$

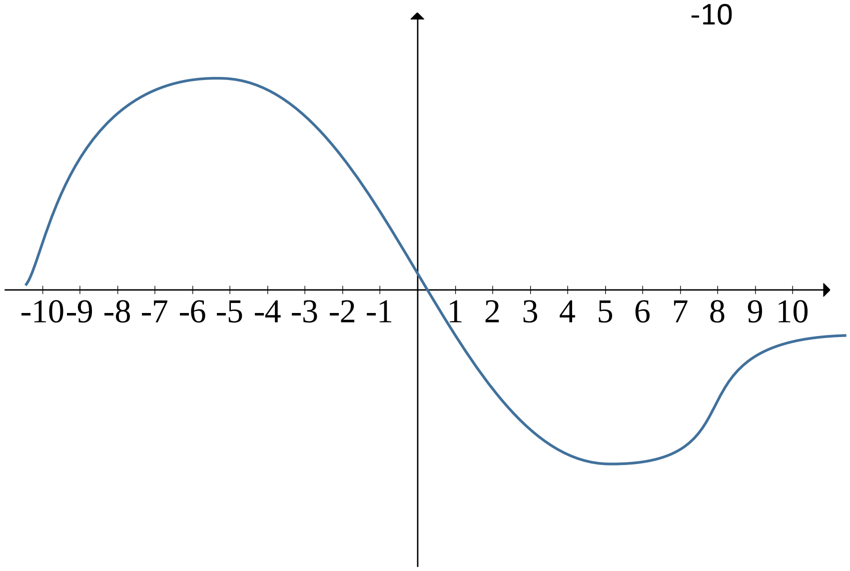
Derivative
$V' = 3l^2 l'$

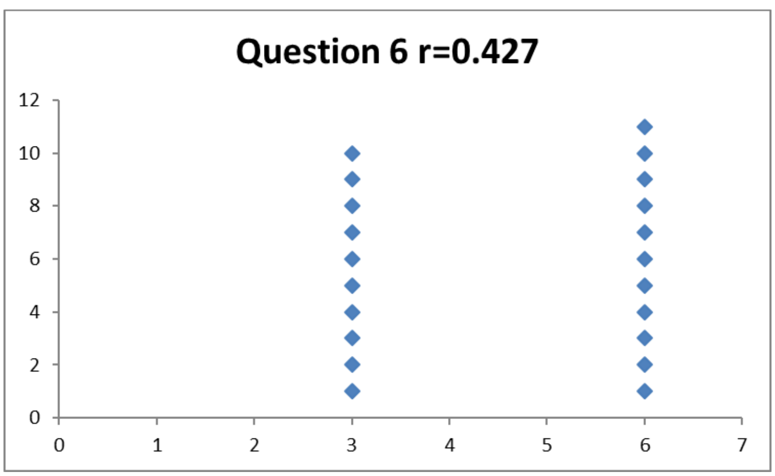
Plug in and solve
$3 = 3(4^2)l'$
$l' = \frac{1}{16}$



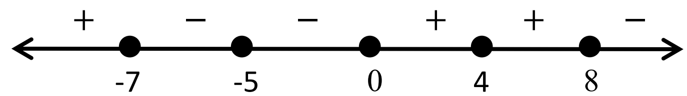
6) Graph a continuous and differentiable function that has the sign chart below with inflection points at $x = 0$ and $x = 7.5$. (6 points

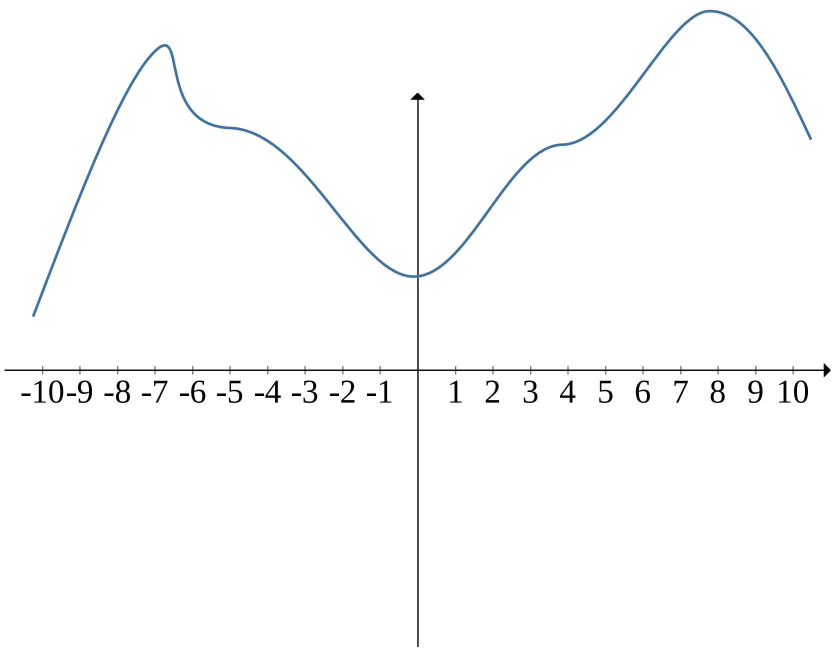


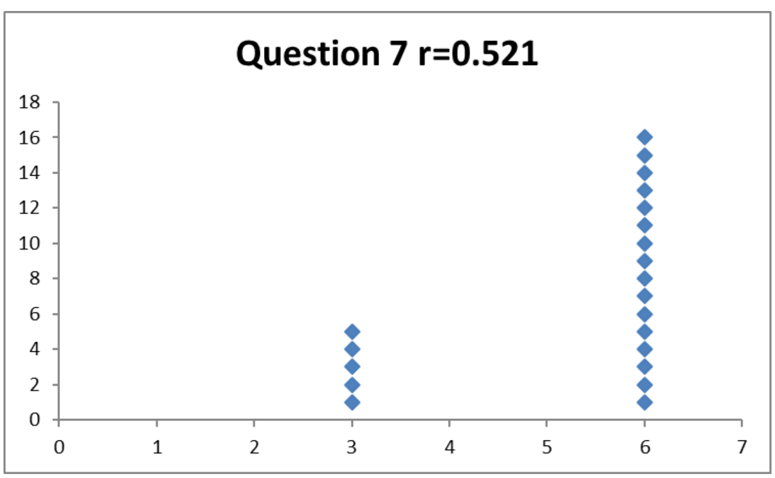




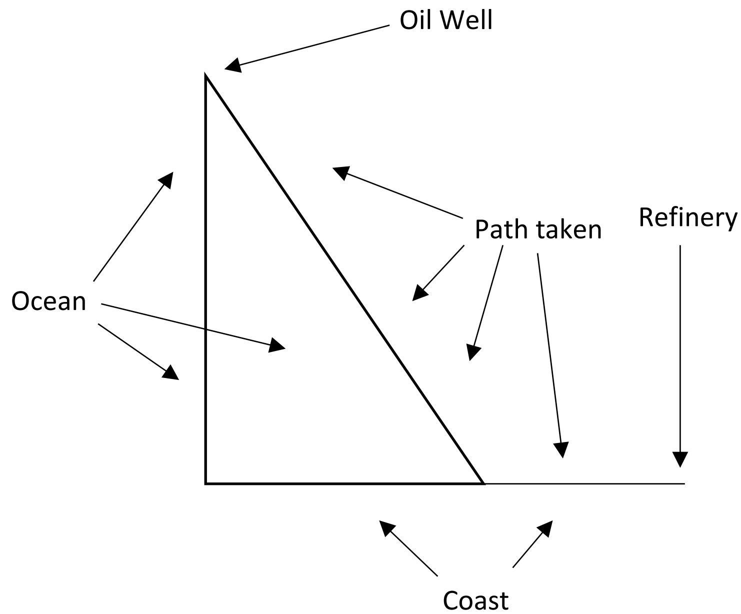
7) Graph a continuous and differentiable function that has the sign chart given below. (6 points)







8) An offshore oil well is 5 kilometers off the coast. The refinery is 6 kilometers down the coast. Laying pipe in the ocean is twice as expensive as on land. How much of the path taken should be on land in order to minimize the cost? (6 points)



Answer: $\boxed{\frac{5}{\sqrt{3}}km}$

Equation
$C = 2h + (6 - x)$

Relatinoship
$5^2 + x^2 = h^2$
$h = \sqrt{5^2 + x^2}$

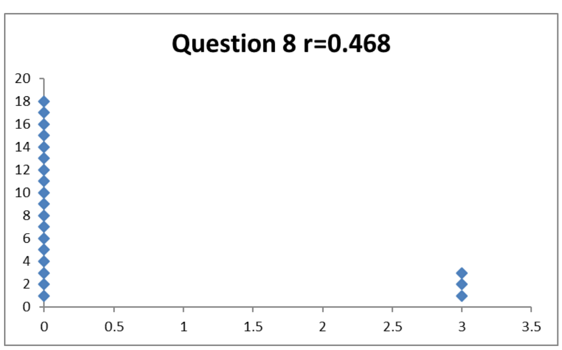
Single variable Equation
$C = 2\sqrt{5^2 + x^2} + (6 - x)$

Derivative
$C' = \frac{2x}{\sqrt{5^2+x^2}} - 1$

$\frac{2x}{\sqrt{5^2+x^2}} - 1 = 0$
$\frac{2x}{\sqrt{5^2+x^2}} = 1$

$2x = \sqrt{5^2 + x^2}$
$4x^2 = 25 + x^2$
$3x^2 = 25$
$x^2 = \frac{25}{3}$
$x = \sqrt{\frac{25}{3}} = \frac{5}{\sqrt{3}}$

(& Check on a sign chart that it's actually a min!)

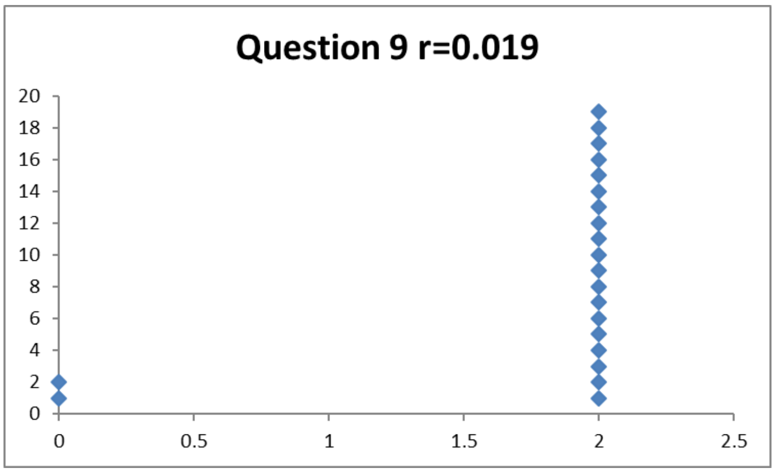


Grading note: This problem was in the homework. ☹

Part 2: Conceptual Understanding

9) Which theorem or mathematical concept *best* embodies the idea that when considering differentiable functions, instantaneous change must sometimes be equal to average rate of change? (2 points)

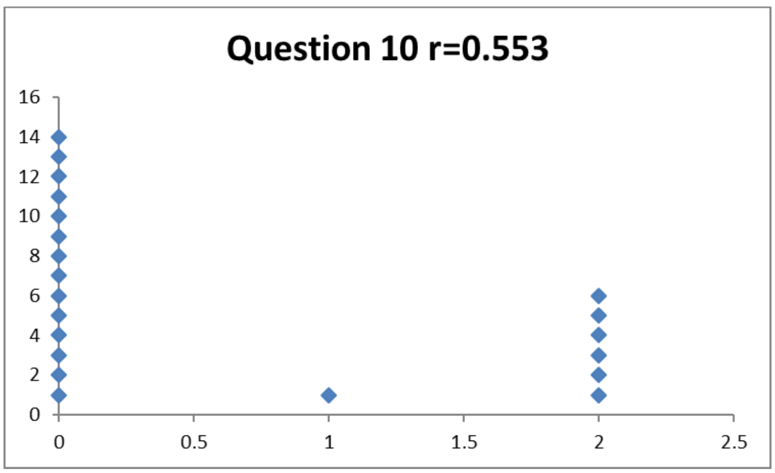
(A) A Limit
(B) A Derivative
(C) Piecewise Functions
(D) The Squeeze Theorem
(E) The Intermediate Value Theorem
(F) Rolle's Theorem
(G) The Mean Value Theorem
(H) The Extreme Value Theorem



10) Which theorem or mathematical concept *best* embodies the idea that some functions make a sort of instantaneous jump where they had one type of y-values and suddenly they have much larger or smaller y-values? (2 points)

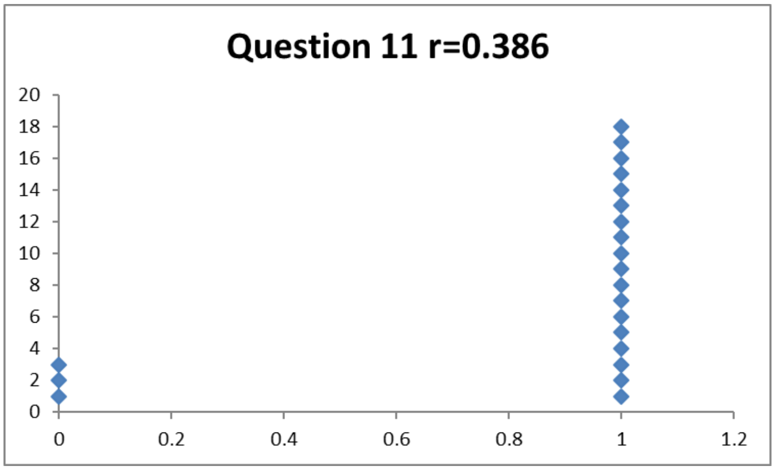
(A) A Limit
(B) A Derivative
(C) Piecewise Functions
(D) The Squeeze Theorem
(E) The Intermediate Value Theorem
(F) Rolle's Theorem
(G) The Mean Value Theorem
(H) The Extreme Value Theorem

Half credit: A Limit, because those also sometimes illustrate jumps.



11) Calculus 1 involves three big concepts. We've covered two of them so far, with the third to start after spring break. What are the two big concepts we've covered so far? (1 point)
(Circle two items)

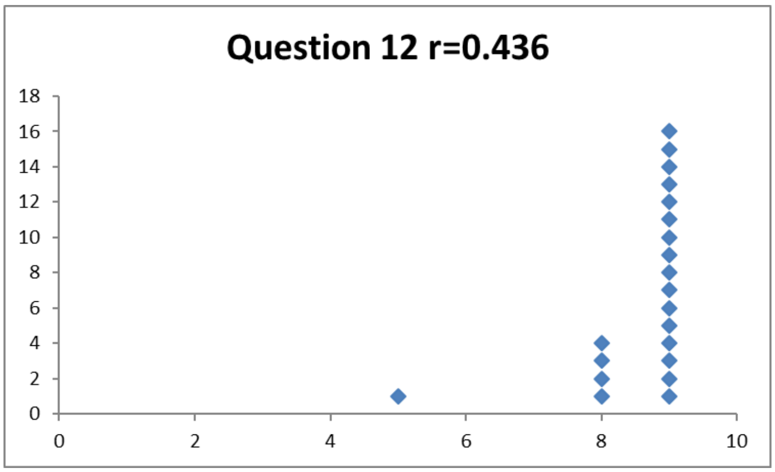
(A) Limit
(B) Infinity
(C) Derivatives
(D) Related Rates



12) Find the antiderivative of $f(x) = 3x^2 + 4x$ such that $F(1) = 5$. Simplify and show your work.
(9 points)

$$F(x) = \frac{3x^3}{3} + \frac{4x^2}{2} + C = x^3 + 2x^2 + C$$
$$5 = 1 + 2 + C$$
$$C = 2$$

$$F(x) = x^3 + 2x^2 + 2$$



13) Find the maximum of $f(x) = 2x^3 + 3x^2 - 36x$ using the second-derivative test. Show your work.
(9 points)

$$f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x + 3)(x - 2)$$

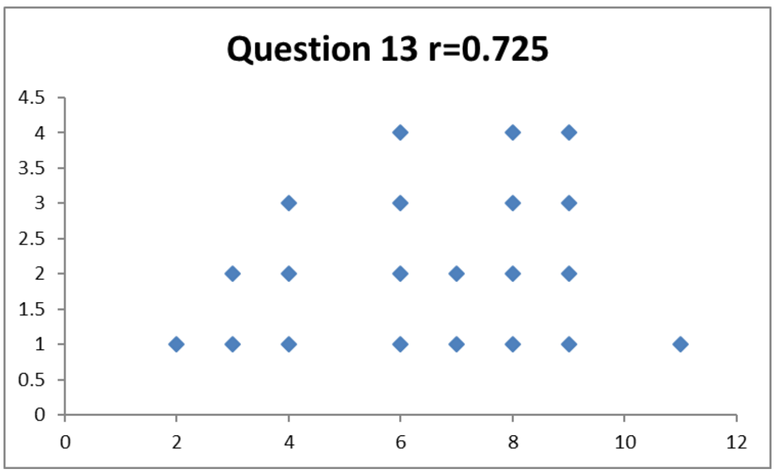
CVs: $x = -3, +2$

$$f''(x) = 12x + 6$$

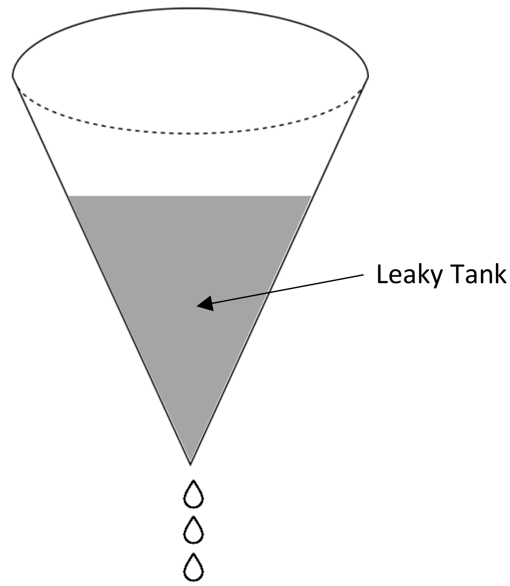
$f''(2) = 24 + 6 = 30$, concave up means min.

$f''(-3) = -36 + 6 = -30$, concave down means max.

$$f(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3) = -54 + 27 + 108 = 81$$



14) An inverted conical tank is used to hold gasoline. However, unfortunately it leaks. The height of the tank is four times the radius of the cone. When the height of the gas is 4 feet and decreasing at a rate of 1ft/min, how quickly is the gas leaking out the hole in the bottom?
(9 points)



Equation

$$V = \frac{1}{3}\pi r^2 h$$

(1 point)

Variables

$h = 4$
$h' = -1$
$V = ?$
$V' = ???$

(1 point)

Similar Triangles/Aspect ratio/Proportion

$h = 4r$

(2 points)

Simplify & Differentiate

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h = \frac{\pi h^3}{3 \cdot 4^2}$$

(1 point)

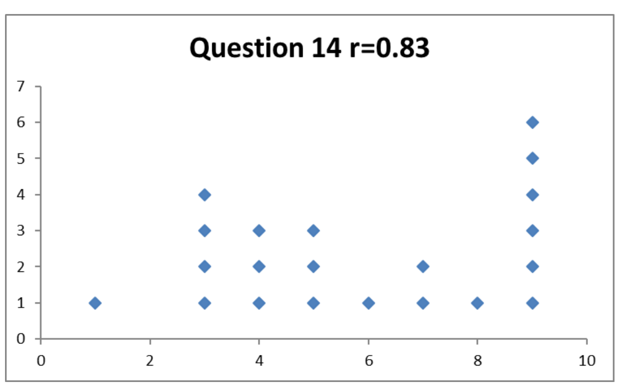
$$V' = \frac{3\pi h^2 h'}{3 \cdot 4^2} = \frac{\pi h^2 h'}{4^2}$$

(2 points)

Solution

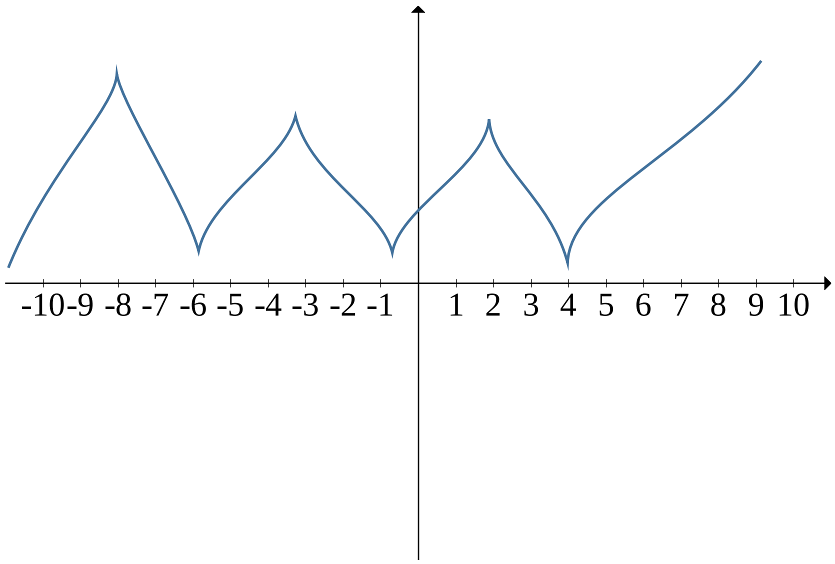
$V' = \frac{\pi 4^2(-1)}{4^2} = -\pi \text{ ft}^3/\text{min}$ (But positive because it's leaking *out* the hole)

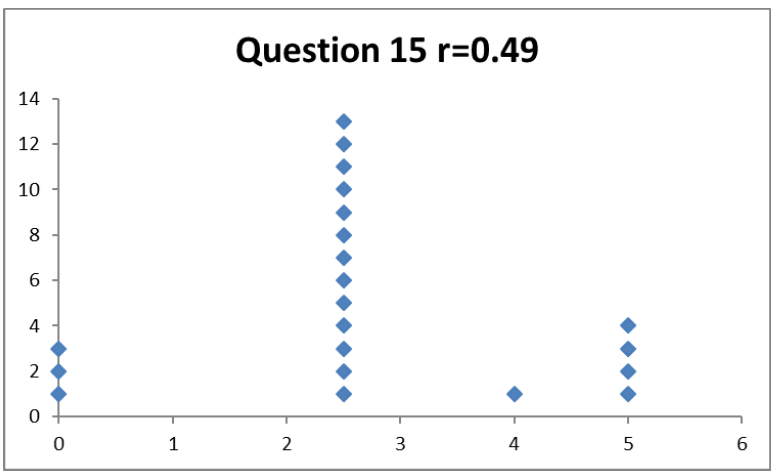
(1 point - answer)
(1 point - units)



Part 3: Applications

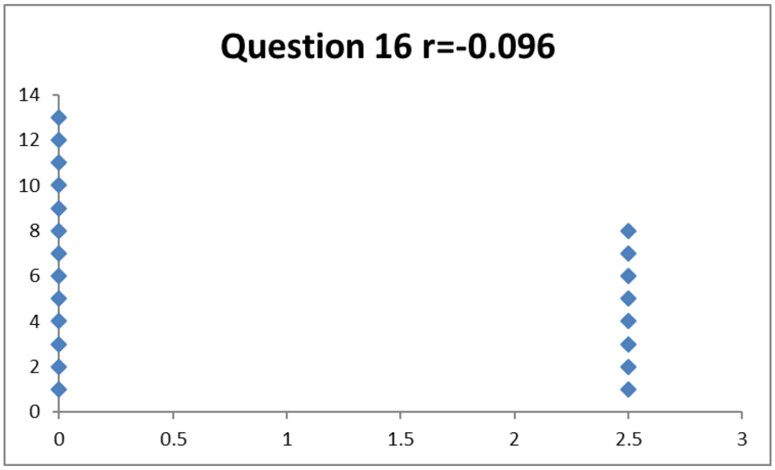
15) Draw an example of a continuous function that has 3 local maxima, but whose derivative is never zero.





16) Let $f(t)$ be a position function. The object it represents is accelerating, but accelerating less quickly over time. For a brief instantaneous moment while accelcerating, its position pauses. Which of the following apply at this moment? Circle all that apply. (5 points)
(Circle three items)

(A) $f(t) > 0$
(B) $f(t) < 0$
(C) $f(t) = 0$
(D) $f'(t) > 0$
(E) $f'(t) < 0$
(F) $f'(t) = 0$
(G) $f''(t) > 0$
(H) $f''(t) < 0$
(I) $f''(t) = 0$
(J) $f'''(t) > 0$
(K) $f'''(t) < 0$
(L) $f'''(t) = 0$



Grading Note: I threw this question out, meaning the test is considered to be out of 95 points instead of 100 points. Why? I would not throw out a question lightly, *especially in a sequence course*. On one hand, I actually had high hopes for this question and thought a number of people would be able to figure it out. On the other hand, it has a negative (albeit small) item analysis correlation which set off some red flags for me, causing me to reflect on this problem to figure out what happened.

Most poignantly, I noticed that several people circled multiple answers in the same column. This is obviously a terrible contradiction because a function cannot be positive and negative simultaneously.

So I think it was not the content of the question, but rather the phrasing of the choices. Perhaps it would have been better to break it up into three questions to piece out each individual idea below.

Here is the reasoning you should be able to go through to figure out the question as stated:

1. The object is accelerating means $f''(t) > 0$
2. …but less quickly. That's the derivative of acceleration, so $f'''(t) < 0$.
3. The position pauses. That means position stops changing, which is the derivative of position. So $f'(t) = 0$.

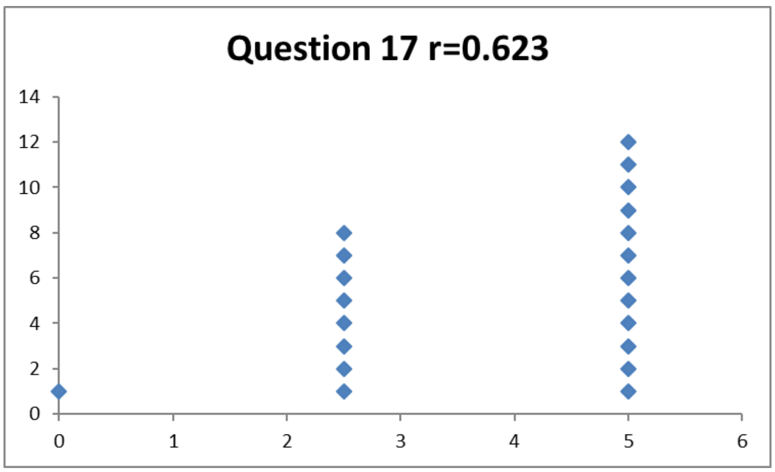
Part 4: Review

17) Find the derivative of the function below. (5 points)

$$f(x) = xe^{2x}$$

Answer: $\boxed{e^{2x} + 2xe^{2x}}$

$$f'(x) = e^{2x} + xe^{2x} \cdot 2$$



Grading Note: This is a very simple question on tools we use practically every day in calculus. If you didn't easily get full credit, you should re-evaluate your study method because you've completely missed some of the fundamentals.

Studying should be like a pyramid: most of your time is at the bottom mastering fundamentals. Once you know those, work your way up to more difficult and nuanced concepts.

I've seen people spend most of their time spinning their wheels on the hardest problems when they should be spending most of their time mastering the fundamentals.

18) Find the limit below. (5 points)

$$\lim_{x \to 4^+} \frac{x-5}{(x-2)(x-4)}$$

Answer: $\boxed{-\infty}$