Name ____ Test 2, Spring 2019

1) Determine whether the series below converges or diverges. Circle which test(s) you use. (6 points)

[Divergence Test] [Integral Test] [Comparison Test] [Limit Comparison Test] [Ratio Test] [Root Test] [Geometric Series] [p-Series] [Alternating Series]

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$$

$$(1) \lim_{k\to\infty} \frac{(-1)^k}{k^3} = 0$$

$$(2) \frac{1}{(k+1)^3} < \frac{1}{k^3}$$

By the alternating series test we see that from 1 and 2 above, the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$ converges.

2) Determine whether the series below converges or diverges. Circle which test(s) you use. (6 points)

[Divergence Test] [Integral Test] [Comparison Test] [Limit Comparison Test] [Ratio Test] [Root Test] [Geometric Series] [p-Series] [Alternating Series]

$$\sum_{k=1}^{\infty} \frac{1}{k^2+4} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

By the comparison we see from the above that the series $\sum_{k=1}^{\infty} \frac{1}{k^2+4}$ converges.

3) Find the 3rd order Taylor Polynomial of $f(x) = e^{2x}$ centered at $x = 1$.

(6 points)

		At $x = 1$?	Coef of
$f(x)$	e^{2x}	e^2	x^0
$f'(x)$	$2e^{2x}$	$2e^2$	x^1
$f''(x)$	$4e^{2x}$	$4e^2$	x^2
$f'''(x)$	$8e^{2x}$	$8e^2$	x^3

$$e^{2x} \approx e^2 + 2e^2(x-1) + \frac{4e^2(x-1)^2}{2} + \frac{8e^2(x-1)^3}{3!}$$

4) Use your Taylor Polynomial from the previous problem to write down a formula that approximates the number $e^{2.2}$

(6 points)

$$e^{2.2} = e^{2(1.1)} \approx e^2 + 2e^2(1.1-1) + \frac{4e^2(1.1-1)^2}{2} + \frac{8e^2(1.1-1)^3}{3!}$$

5) How accurate is your approximation from the previous problem? Note that the remainder in a n^{th} order Taylor Polynomial is bounded by $\frac{M(x-a)^{n+1}}{(n+1)!}$.

(6 points)

$$\frac{16e^{2.2}(1.1-1)^4}{4!}$$

6) Complete ONE of the following problems.

(a) Find the radius of convergence of the Taylor series below. (8 points maximum)

$$\sum_{k=1}^{\infty}(2x)^k$$

(b) Find the interval of convergence of the Taylor series below. (10 points – full credit)

$$\sum_{k=1}^{\infty}2(x-5)^k$$

[Word crashed and we lost these solutions]

7) Find the series expansion for ONE of the functions below, centered at $x = 0$. Use any method.

(a) $f(x) = \cosh(x)$
(8 points maximum)

(b) $f(x) = e^{x^2}$
(10 points – full credit)

….math we missed

$$e^b = 1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} + \cdots$$

It looks we want the series with $b = x^2$. This gives us:

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

8) Find the series expansion for ONE of the functions below, centered at $x = 0$. Use any method.

(a) $f(x) = \frac{1}{(1-x)^2}$
(8 points maximum)

Note the geometric series:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

Take a derivative:

$$-\frac{1}{(1-x)^2}(-1) = \sum_{k=1}^{\infty} kx^{k-1}$$
$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1}$$

(b) $f(x) = \frac{1}{(2-x)^2}$
(10 points – full credit)

Note the geometric series:

$$\frac{1}{2-x} = \frac{1}{2} \cdot \frac{1}{1-\left(\frac{x}{2}\right)} = \frac{1}{2}\sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k$$

Take a derivative:

$$-\frac{1}{2}\frac{1}{\left(1-\left(\frac{x}{2}\right)\right)^2}\left(-\frac{1}{2}\right) = \frac{1}{4}\sum_{k=1}^{\infty} k\left(\frac{x}{2}\right)^{k-1}$$
$$\frac{1}{4}\frac{1}{\left(1-\left(\frac{x}{2}\right)\right)^2} = \frac{1}{4}\sum_{k=1}^{\infty} k\left(\frac{x}{2}\right)^{k-1}$$
$$\frac{1}{(2-x)^2} = \frac{1}{4}\sum_{k=1}^{\infty} k\left(\frac{x}{2}\right)^{k-1}$$

9) Given the parametric equations below, find a single equation that describes the same curve. (6 points)

$$x = t^2 + 2$$
$$y = 4t$$

$$\frac{y}{4} = t$$

$$x = \left(\frac{y}{4}\right)^2 + 2$$

10) Given the parametric equations below, find the slope of the tangent line at $t = 2$. (8 points)

$$x = 3 - t$$
$$y = 3t^2 + 12$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t}{-1} = -6t$$

At $t = 2$ we get:

$$m = -12$$

11) Find the equation of the line tangent to the curve in the previous problem at $t = 2$. (6 points)

$$x = 3 - 2 = 1$$
$$y = 3(4) + 12 = 24$$

$$y - 24 = (-12)(x - 1)$$
$$y = -12x + 36$$

12) On the axis provided, graph ONE of the polar curves below.

(a) $r = 3\sin(2\theta)$ for $0 \leq \theta \leq \frac{\pi}{2}$.

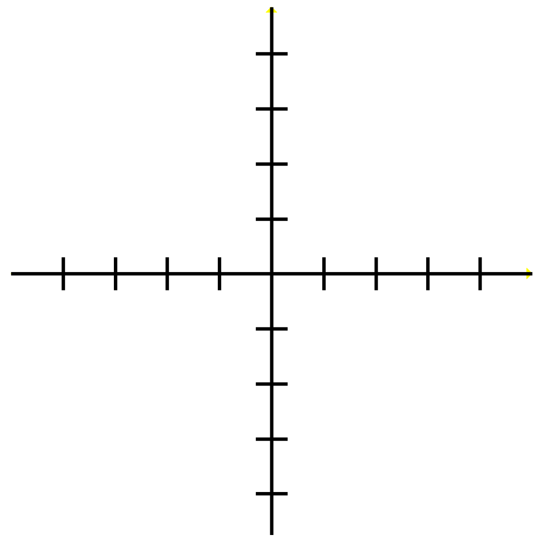
(8 points – full credit)

(b) $r = 3\cos(3\theta)$ for $0 \leq \theta \leq \frac{\pi}{6}$.

(8 points – full credit)

(c) $r = 3\cos(2\theta)$ for $0 \leq \theta \leq \pi$.

(6 points maximum)



13) Complete ONE of the following problems.

(a) Given a polar equation $r = f(\theta)$, what is the formula for $\frac{dy}{dx}$ in terms of r and θ?

(4 points maximum)

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$$

(b) Find the slope of the line tangent to $r = \cos(3\theta)$ at $\frac{\pi}{6}$.

(6 points – full credit)

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} = \frac{(-3\sin(3\theta))\sin(\theta) + \cos(3\theta)\cos(\theta)}{(-3\sin(3\theta))\cos(\theta) - \cos(3\theta)\sin(\theta)}$$

$$= \frac{\frac{(-3)}{2} + 0}{(-3)\frac{\sqrt{3}}{2} - 0} = \frac{1}{\sqrt{3}}$$

$$\cos(3\theta) = \cos\left(\frac{\pi}{2}\right) = 0$$

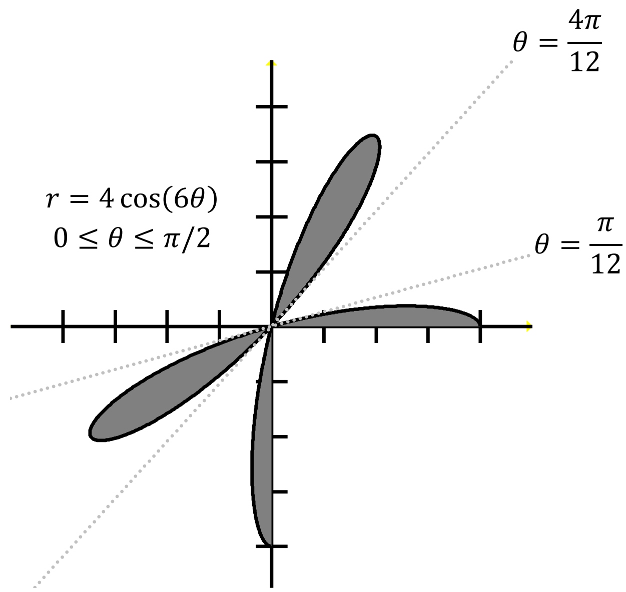
$$\sin(3\theta) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

14) Write down an integral formula for the area shown in the graph below. You do not need to calculate it. (6 points)

$\int_a^b \frac{1}{2} r^2 d\theta = \int_0^{\frac{\pi}{12}} \frac{1}{2} (4\cos(6\theta))^2 d\theta$



15) **2 point bonus:** Use Taylor series to find the limit below.

$$\lim_{x\to 0}\frac{e^x-1}{x}$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

$$\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0}\frac{1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots - 1}{x} = \lim_{x\to 0}\frac{x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots}{x}$$

$$= \lim_{x\to 0}\frac{1 + \frac{x^1}{2} + \frac{x^2}{3!} + \cdots}{1} = 1$$

16) **10 points bonus**: Write the series below as a summation in sigma notation.

(Warning: Don't waste all you're time thinking about this problem. It's a huge bonus for a reason – it's hard and I don't expect many people to figure it out)

$$2 + 4 - 8 - 16 + 32 + 64 - 128 - 256 + 512 + 1024 - 2048 - 4096 + \cdots$$

$$\sum_{k=1}^{\infty} 2^k\left(\sin\left(\frac{\pi k}{2}\right) + \cos\left(\frac{\pi k}{2}\right)\right)$$