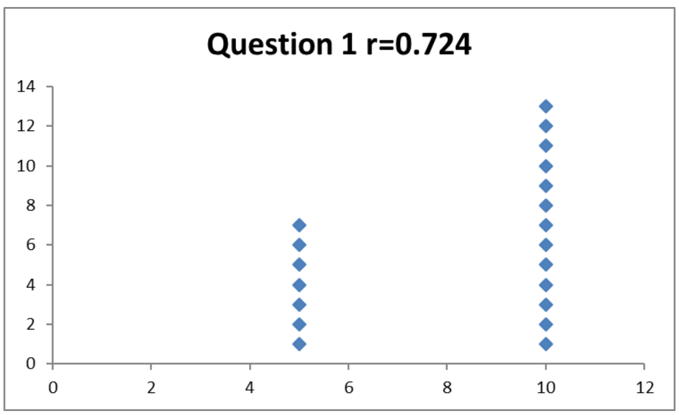
Name ____ Test 1, Spring 2020

Part 1: Computational Skills

1) Find the integral below. (10 points)

$$\int_0^1 \frac{1}{\sqrt[3]{x}}dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt[3]{x}}dx = \lim_{t\to 0^+}\int_t^1 (x)^{-\frac{1}{3}}dx = \lim_{t\to 0^+}\left(\frac{x^{\frac{2}{3}}}{\frac{2}{3}}\Bigg|_t^1\right) = \lim_{t\to 0^+}\left(\frac{3}{2}-\frac{3}{2}t^{\frac{2}{3}}\right) = \frac{3}{2}$$

***Common mistake: Not having the limit. It's an improper integral, which requires a limit. This was costly pointwise because it's a big conceptual mistake, not merely algebra or forgetting an equals sign.



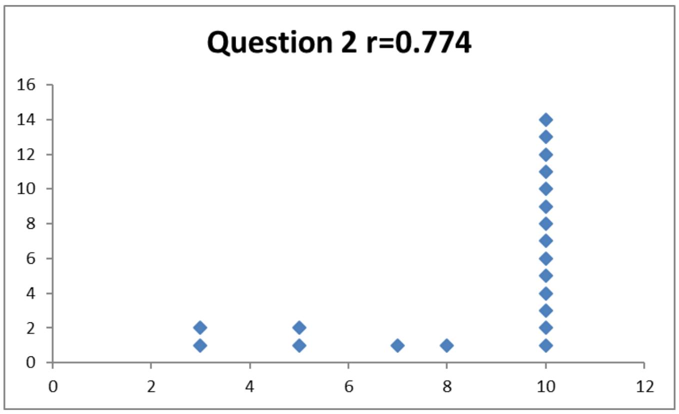
2) Find the integral below. (10 points)

$$\int \sin^3(5x)\,dx = \int \sin^2(5x)\sin(5x)\,dx = \int (1-\cos^2(5x))\sin(5x)\,dx = -\frac{1}{5}\int (1-u^2)du$$

$$-\frac{1}{5}\left(u-\frac{u^3}{3}\right)+C = -\frac{1}{5}\left(\cos(5x)-\frac{\cos^3(5x)}{3}\right)+C$$

$u = \cos(5x)$

$du = -5\sin(5x)\,dx$

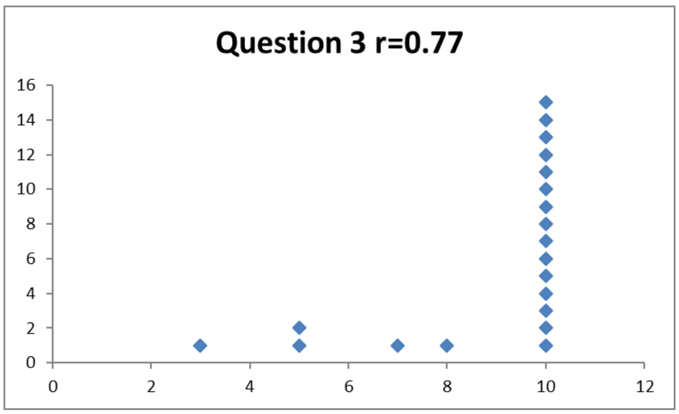


3) Find the integral below. (10 points)

$$\int xe^{4x}dx = \frac{1}{4}xe^{4x} - \frac{1}{4}\int e^{4x}dx = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x}dx + C$$

$u = x$ $\qquad dv = e^{4x}dx$

$du = dx$ $\qquad v = \frac{1}{4}e^{4x}$



4) Find the integral below. (10 points)

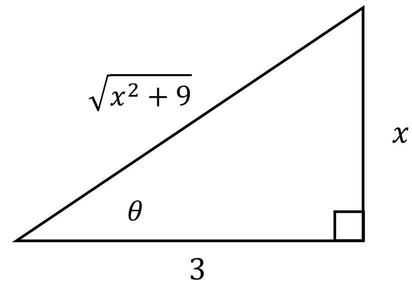
$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \sin(\theta)\,(3\tan(\theta))^2 3\sec^2(\theta)\,d\theta$$

$$= 27\int \sin(\theta)\frac{\sin^2(\theta)}{\cos^2(\theta)}\frac{1}{\cos^2(\theta)}\,d\theta = 27\int \frac{1-\cos^2(\theta)}{\cos^4(\theta)}\sin(\theta)\,d\theta$$

$$= -27\int \frac{1-u^2}{u^4}\,du = -27\int u^{-4}-u^{-2}\,du$$

$$= -\frac{27u^{-3}}{-3}+\frac{27u^{-1}}{-1}+C = \frac{9}{\cos^3(\theta)}-\frac{27}{\cos(\theta)}+C$$

$$= 9\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - \frac{27\sqrt{x^2+9}}{3}+C$$



$$\sin(\theta) = \frac{x}{\sqrt{x^2+9}}$$

$$\cos(\theta) = \frac{3}{\sqrt{x^2+9}}$$

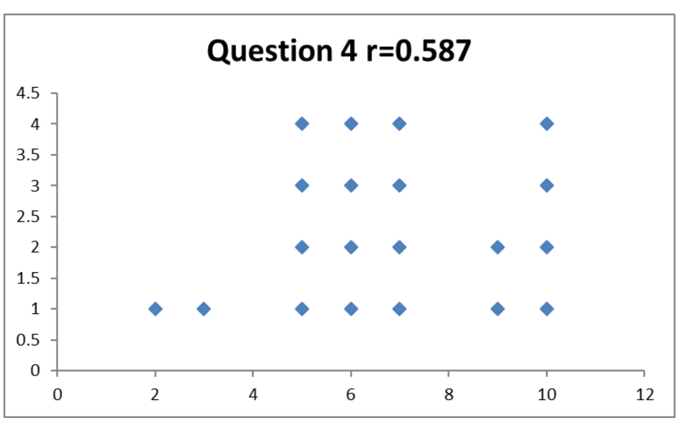
$$\tan(\theta) = \frac{x}{3}$$

$$\sec(\theta) = \frac{\sqrt{x^2+9}}{3}$$

$$\csc(\theta) = \frac{\sqrt{x^2+9}}{x}$$

$$\cot(\theta) = \frac{3}{x}$$

$$\sec^2(\theta)\,d\theta = \frac{1}{3}dx$$



5) Find the integral below. (10 points)

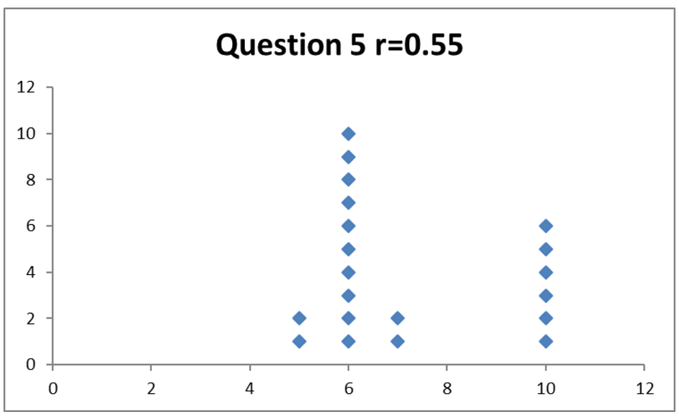
$$\int \frac{4x^2+2x-1}{x^3+x^2}dx = \int \frac{4x^2+2x-1}{x^2(x+1)}dx = \int \frac{3}{x}dx - \int \frac{B}{x^2}dx + \int \frac{1}{x+1}dx$$

$$= 3\ln|x| + \frac{1}{x} + \ln|x+1| + C$$

$$\frac{4x^2+2x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{D}{x+1}$$

$$4x^2+2x-1 = A(x)(x+1) + B(x+1) + Dx^2$$

When $x = 0$: $-1 = B$
When $x = -1$: $4 - 2 - 1 = D; D = 1$
$[x]^2$ coefficient: $4 = A + D; A = 3$



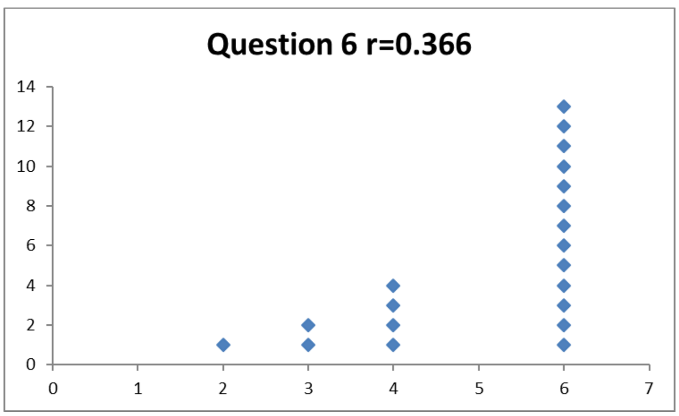
Method	Summation	Maximum Error
Simpsons (Parabolas)	$\frac{\Delta x}{3}\big(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+4f(x_{n-1})+f(x_n)\big)$	$\frac{M_4(b-a)^5}{180n^4}$

Figure 1: Table for the formulas for Simpson's Rule given in class

x	$f(x)$
0	4
2	1
4	3
6	2
8	5

6) Use Simpson's Rule to find an approximation for the integral $\int_0^8 f(x)$, using the information given in the table. (6 points)

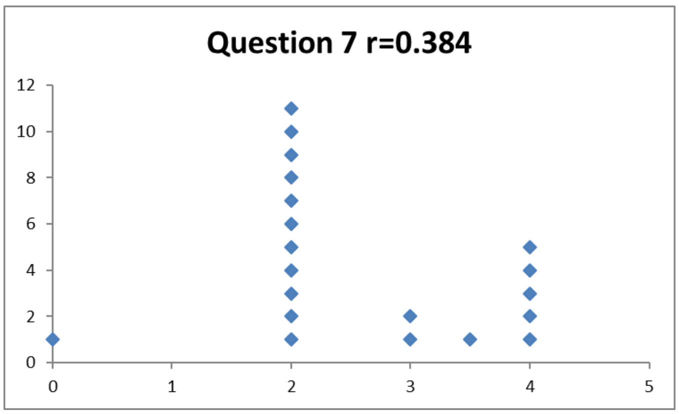
$$\frac{2}{3}(4+4\cdot 1+2\cdot 3+4\cdot 2+5)=\frac{2}{3}(4+4+6+8+5)=\frac{2}{3}(27)=18$$



7) Assume that the absolute value of the fourth derivative of the function $f(x)$ is at most 4. That is, $M_4 = 4$. Use this information along with your approximation in the previous question to find the largest possible value of the actual answer to the integral. (4 points)

(You may leave your answer as a formula. The arithmetic will be messy; just leave it totally unsimplified)

$$18 + [error] = 18 + \frac{4 \cdot 8^5}{180 \cdot 4^4}$$

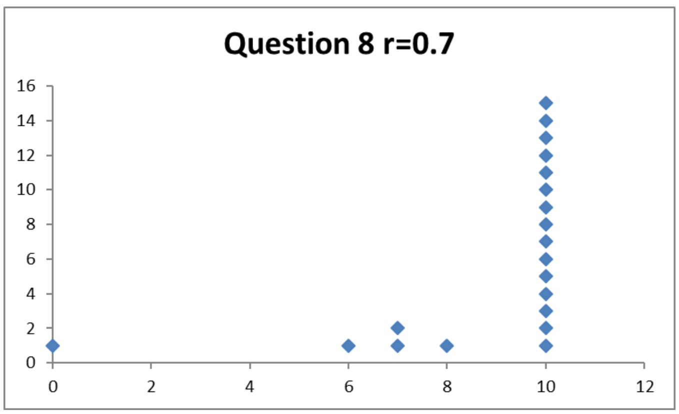


8) Use the formula given in the box here to find the integral below.
(10 points)

$$\int \frac{du}{u\sqrt{2au - u^2}} = \frac{\sqrt{2au - u^2}}{au} + C$$

$$\int \frac{4}{5x\sqrt{6x - x^2}}dx = \frac{4}{5}\int \frac{1}{x\sqrt{6x - x^2}}dx = \frac{4}{5}\frac{\sqrt{6x - x^2}}{3x} + C$$

$$a = 3; u = x$$



Part 2: Conceptual Understanding

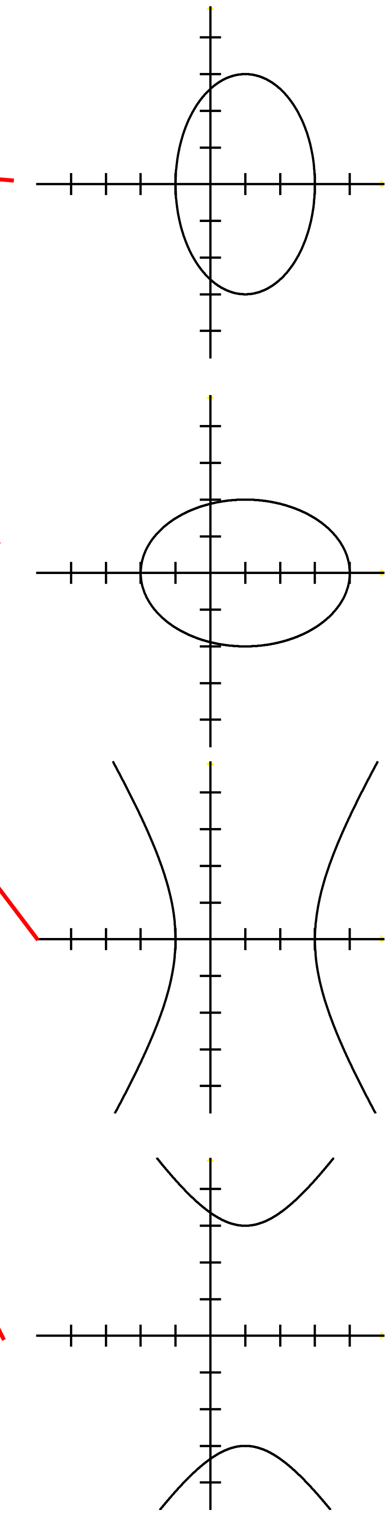
9) Match the four equations below with the four graphs below.
(2 points each)

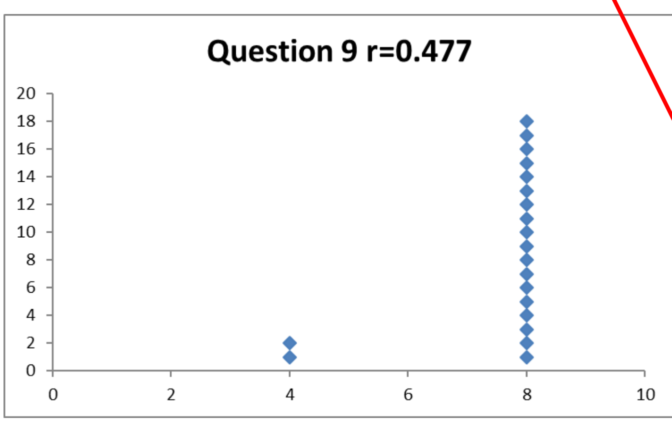
$A)\ \frac{(x-1)^2}{4}+\frac{y^2}{9}=1$

$B)\ \frac{(x-1)^2}{4}-\frac{y^2}{9}=1$

$C)\ -\frac{(x-1)^2}{4}+\frac{y^2}{9}=1$

$D)\ \frac{(x-1)^2}{9}+\frac{y^2}{4}=1$





10) For each of the integrals below, choose which technique of integration would be the BEST method to approach this integral. Place your answer in the blank to the left of the problem. (1 point each)

(A) Standard u-substitution from calculus 1
(B) Integration by parts
(C) Trigonometric Integration (Apply trig formulas, then do other stuff)
(D) Trigonometric substitution
(E) Separation of Partial Fractions
(F) Numerical Integration
(G) Improper Integration

__E__ i. $\int \frac{x^4+3x^3-2x+1}{(x-1)(x^2+1)(x^2-1)} dx$

__B__ ii. $\int xe^{1-x} dx$

__A__ iii. $\int \sin(x) \cos^4(x)\, dx$

__D__ iv. $\int -\frac{12}{x^2\sqrt{4-x^2}} dx$

__B__ v. $\int x \tan^{-1}(2x)\, dx$

__C__ vi. $\int \cos^3(\pi x - 1)\, dx$

__D__ vii. $\int x^2\sqrt{x^3 - 27} dx$

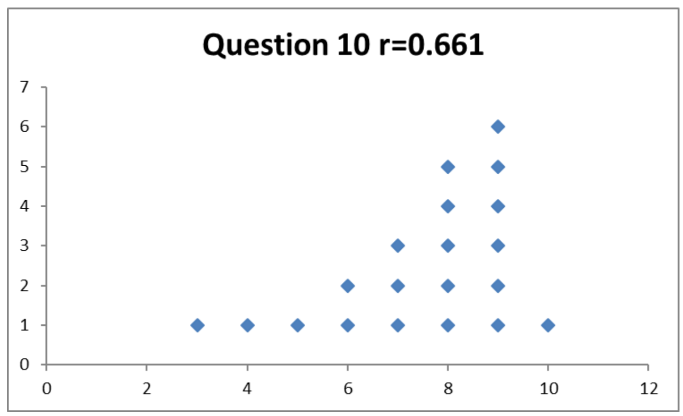
__E__ viii. $\int \frac{x-8}{x^2-x-6} dx$

__D__ ix. $\int \frac{(9-4x^2)^{\frac{5}{2}}}{5x^2} dx$

__E__ x. $\int_0^1 \frac{2x}{x^2-4} dx$

__G__ xi. $\int_1^2 \frac{7}{x-2} dx$

__G__ xii. $\int_1^\infty \frac{1}{\sqrt[4]{x}} dx$



In case you need this …

$$\int \frac{4}{(x^2+1)^2}dx = \frac{2}{x^2+1} + 2\tan^{-1}x$$

Part 3: Applications

11) A tank is to be cut by a "CNC" machine that uses mathematical equations to guide its movement. In particular, it is given the equation below, between $x = 0$ and $x = 3$, and told to rotate it around the x-axis. What is the volume of the tank? (10 points)

$$y = \frac{2x}{x^2+1}$$

The volume is: $\int_0^3 \pi r^2 dx = \pi \int_0^3 \left(\frac{2x}{x^2+1}\right)^2 dx$

$$\int \left(\frac{2x}{x^2+1}\right)^2 dx = \int \frac{4}{x^2+1}dx - \int \frac{4}{(x^2+1)^2}dx = 4\tan^{-1}(x) - 2\tan^{-1}(x) - 2\frac{x}{\sqrt{x^2+1}}\frac{1}{\sqrt{x^2+1}}$$
$$= 2\tan^{-1}(x) - \frac{2x}{x^2+1}$$

Now evaluate this between 0 and 3:

$$2\tan^{-1}(3) - \frac{6}{10} - 0 - 0$$

$\frac{4x^2}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

$4x^2 = (Ax+B)(x^2+1) + Dx + E$

$[x^3]$ coefficient: $0 = A$
$[x^2]$ coefficient: $4 = B$
$[x]$ coefficient: $0 = A + D = D$
$[x^0]$ coefficient: $0 = B + E = 4 + E; E = -4$

Partial credit was given at each conceptual step of the problem:

- Use an integral
- Limits of integration at 0 and 3.
- Formula for volume of a solid of revolution: πr^2
- Attempt to use partial fractions to integrate (Or the proper technique for what you have if you're off track)
- Calculate the integral correctly.