Name ____ Test 2, Spring 2020

Part 1: Computational Skills

1) Determine whether the series converges or diverges. Circle which test(s) you use. (10 points)

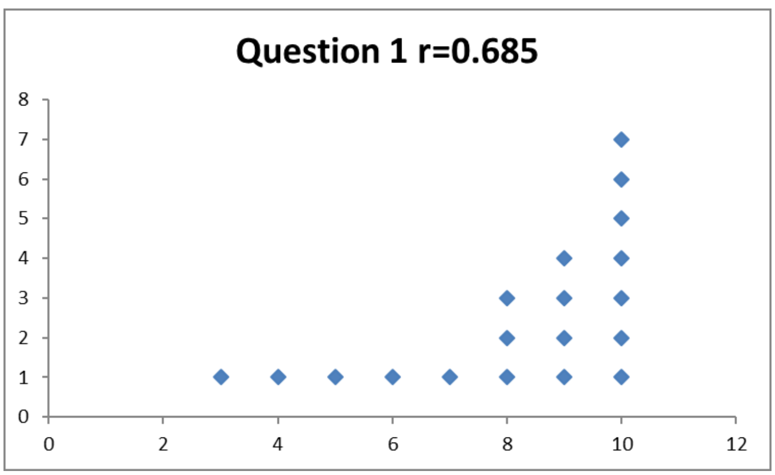
[Divergence Test] [Integral Test] [Comparison Test] [Limit Comparison Test] [Ratio Test] [Root Test] [Geometric Series] [p-Series] [Alternating Series]

$$\sum_{n=1}^{\infty} \frac{3^n + n}{2^n - 1}$$

The best test to use here is probably the divergence test:

$\sum_{n=1}^{\infty} \frac{3^n+n}{2^n-1}$ Diverges because $\lim_{n\to\infty} \frac{3^n+n}{2^n-1} = \infty \neq 0$

Some people also did a comparison test to a geometric series. That works too.



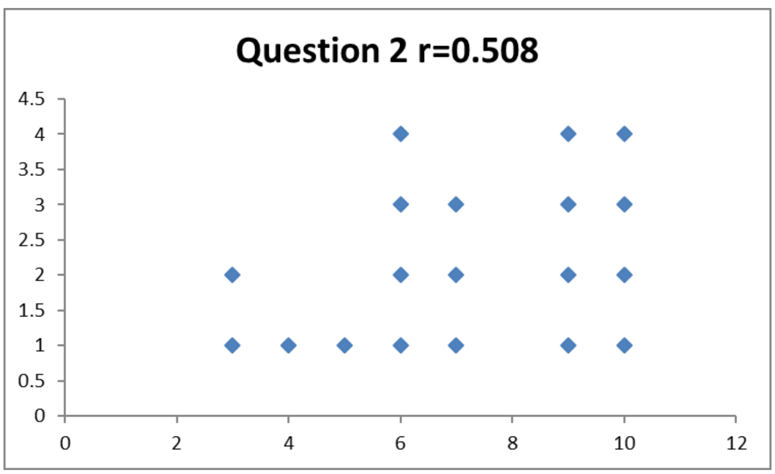
2) Determine whether the series converges or diverges. Circle which test(s) you use. (10 points)
[Divergence Test] [Integral Test] [Comparison Test] [Limit Comparison Test] [Ratio Test] [Root Test] [Geometric Series] [p-Series] [Alternating Series]

$$\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^3}$$

$$\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^3} \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Use a comparison test:

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p series ($p = 3$), and so $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^3}$ converges as well.



3) Determine whether the series converges or diverges. Circle which test(s) you use. (10 points)

[Divergence Test] [Integral Test] [Comparison Test] [Limit Comparison Test] [Ratio Test] [Root Test] [Geometric Series] [p-Series] [Alternating Series]

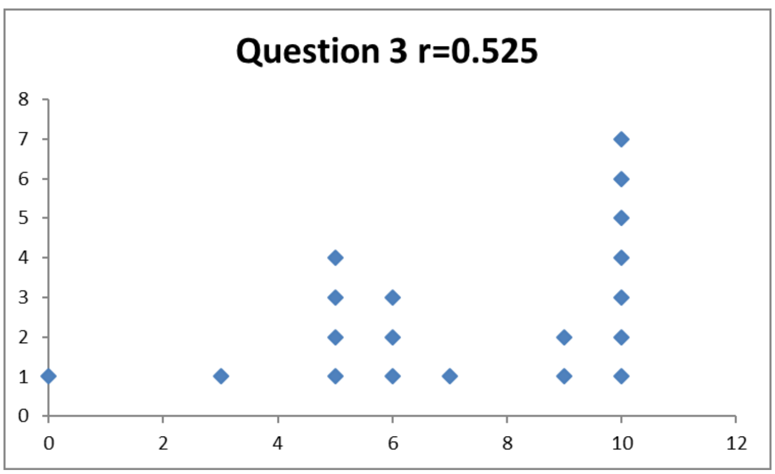
$$\sum_{n=1}^{\infty} \frac{n}{(\ln(n))^n}$$

Ratio test is our best bet here:

$$\lim_{n\to\infty}\left|\frac{\frac{n+1}{(\ln(n+1))^{n+1}}}{\frac{n}{(\ln(n))^n}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{(\ln(n+1))^{n+1}}\cdot\frac{(\ln(n))^n}{n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{(\ln(n))^n}{(\ln(n+1))^{n+1}}\right| = 1\cdot 0 = 0 < 1$$

Hence $\sum_{n=1}^{\infty}\frac{n}{(\ln(n))^n}$ converges.

You could have used root the test as well. Though that's probably a little harder because it involves seeing that $\lim_{n\to\infty}\sqrt[n]{n} = 1$, which is probably more involved than seeing that $\lim_{n\to\infty}\frac{(\ln(n))^n}{(\ln(n+1))^{n+1}} = 0$



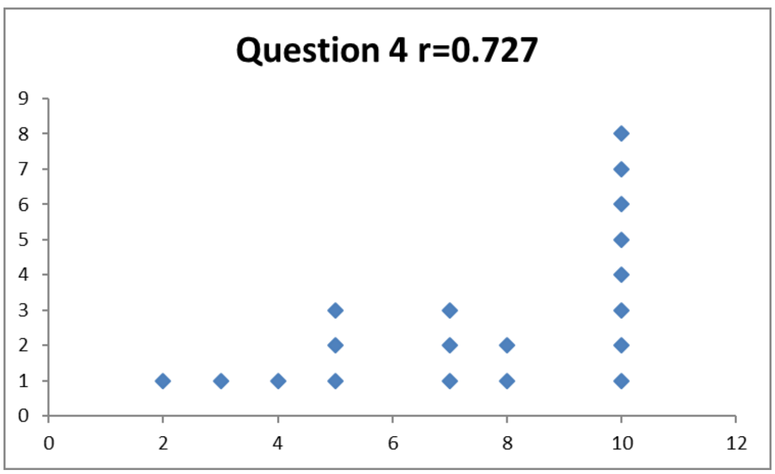
4) Find the radius of the convergence of the power series below. (10 points)

$$\sum_{k=3}^{\infty} \frac{k^2}{5^k}(x-2)^k$$

$$\lim_{n\to\infty}\left|\frac{\frac{(k+1)^2}{5^{k+1}}(x-2)^{k+1}}{\frac{k^2}{5^k}(x-2)^k}\right| = \lim_{n\to\infty}\left|\frac{(k+1)^2(x-2)^{k+1}5^k}{5^{k+1}(x-2)^k k^2}\right| = \lim_{n\to\infty}\left|\frac{(k+1)^2(x-2)}{5k^2}\right| = \frac{|x-2|}{5} < 1$$

$$|x-2| < 5$$

$$R = 5$$



5) Find a power series expansion of the function below, centered at $x = 1$. Write your answer using sigma notation. (10 points)

$$\frac{1}{x^2}$$

Let's do a Taylor series:

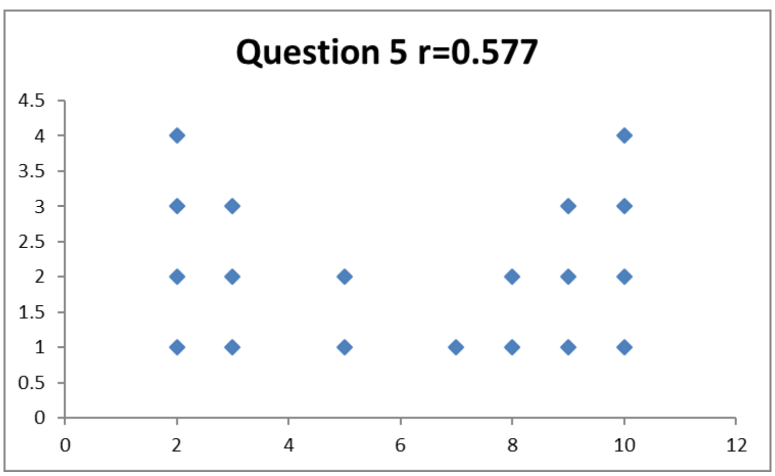
k	$f^{(k)}(x)$	$f^{(k)}(1)$	Whole Term
0	$\frac{1}{x^2} = x^{-2}$	1	1
1	$-2x^{-3}$	-2	$-2(x-1)$
2	$6x^{-4}$	6	$\frac{6(x-1)^2}{2!}$
3	$-24x^{-5}$	-24	$-\frac{24(x-1)^3}{3!}$
4	$120x^{-6}$	120	$\frac{120(x-1)^4}{4!}$

$$\frac{1}{x^2} = 1 - 2(x-1) + \frac{6(x-1)}{2} - \frac{24(x-1)^3}{6} + \frac{120(x-1)^4}{24} + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(k+1)!\,(x-1)^k}{k!} = \sum_{k=0}^{\infty} (-1)^k (k+1)(x-1)^k$$

You could also use the derivative of a geometric series expansion of $\frac{1}{1-(1-x)}$. It probably actually requires less computational work, but more conceptual understanding and algebraic maturity.

That trick doesn't work for $\frac{1}{x^2}$ directly, however, because it won't be centered at $x = 1$.



6) Find a power series expansion of the function below, centered at $x = 0$. (10 points)

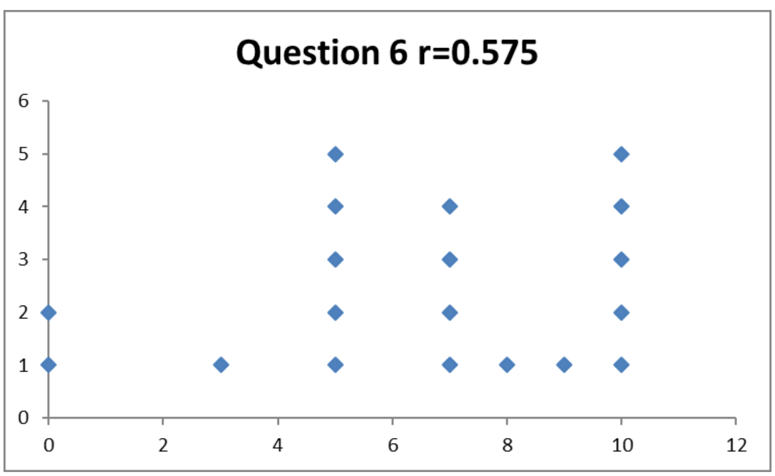
$$\cos(5x)$$

Let's do a Taylor series:

k	$f^{(k)}(x)$	$f^{(k)}(0)$	Whole Term
0	$\cos(5x)$	1	1
1	$-5\sin(5x)$	0	0
2	$-25\cos(5x)$	-25	$-\frac{25x^2}{2!}$
3	$125\sin(5x)$	0	0
4	$625\cos(5x)$	625	$\frac{5^4x^4}{4!}$

$$\cos(5x) = 1 - \frac{25x^2}{2} + \frac{5^4x^4}{4!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{5^{2k}x^{2k}}{(2k)!}$$

If you knew the power series expansion for $\cos(u)$, you could have used $u = 5x$. Just make sure you show how your work.

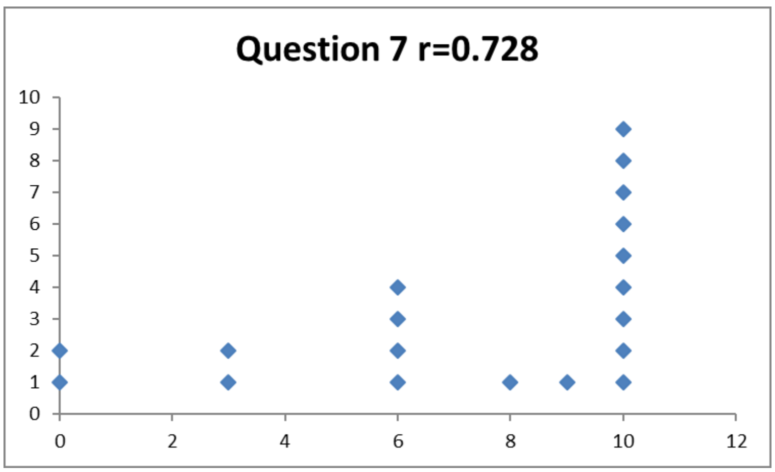


Part 2: Conceptual Understanding

7) Let $f(x)$ be a function satisfying with power series expansion $\sum_{k=0}^{\infty} a_k(x-2)^k$, centered at $x = 2$ with radius of convergence 7. It is known that $f(2) = 5$. What is $\sum_{k=0}^{\infty} a_k(2-2)^k$? (10 points)

$$\sum_{k=0}^{\infty} a_n(2-2)^n = 5$$

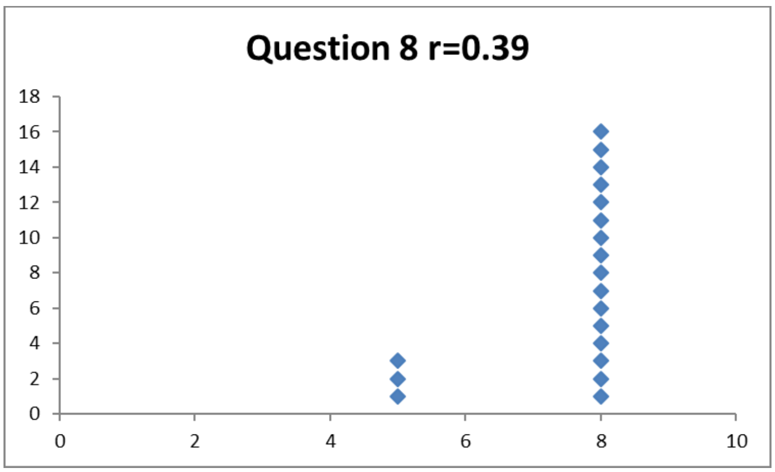
The power series expansion centered at $x = 2$ with radius 7, meaning they are equal on the interval $(-5,9)$. That includes $x = 2$, so $\sum_{k=0}^{\infty} a_n(2-2)^n = f(2) = 5$.
(Yes most of the terms are zero, but a_0 is not)



8) If a power series has radius of convergence 4 and the interval of convergence does not include either endpoint, for how many integer x-values can the series possibly converge? (10 points)

8

If $R = 4$, then that means that the interval has width 8, such as for example $(-4,4)$ or $(0,8)$. Those intervals only contain 7 integers, such as $\{1,2,3,4,5,6,7\}$. However, if it starts at a fraction, it can contain 8 integers, such as $(0.5,8.5)$. This still has radius 4, and center 4.5.



Part 3: Applications

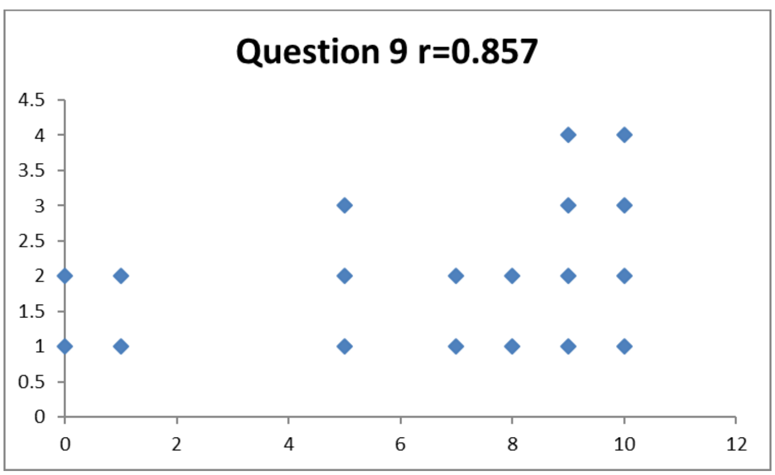
9) Below is a Taylor series (centered at $x = 0$) that represents a function. Suppose we are going to ask a computer to evaluate this function at $x = 1$. The computer uses the first 13 terms. How accurate is the computer's approximation? It is known that f is infinitely differentiable, and each derivative is no larger than 5. (10 points)

$$\sum_{k=0}^{\infty} \frac{(k-3)x^k}{k}$$

The formula for error is $\frac{M(x-0)^{k+1}}{(k+1)!}$, which in this case is:

$$\frac{5(1)^{14}}{14!} = \frac{5}{14!}$$

**Note: This question is a bit messed up in several ways. First, the first 13 terms would include the $k = 0$ term, making the 14th term $k = 13$, so full credit was given for $\frac{5}{13!}$ as well. Also, what's up with $k = 0$? We would get a zero in the denominator, so this power series is impossible. Anybody that noticed that received extra credit.



Part 4: Review

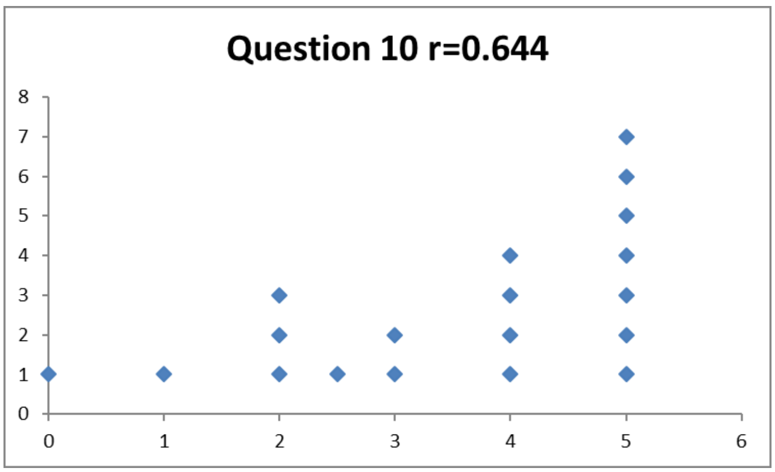
10) Find the integral below. (5 points)

$$\int x\sec^2(x)\,dx$$

$$\int x\sec^2(x)\,dx = x\tan(x) - \int \tan(x)\,dx = x\tan(x) - \int \frac{\sin(x)}{\cos(x)}dx = x\tan(x) + \ln|cos(x)| + C$$

$u = x$ $\quad dv = \sec^2(x)\,dx$

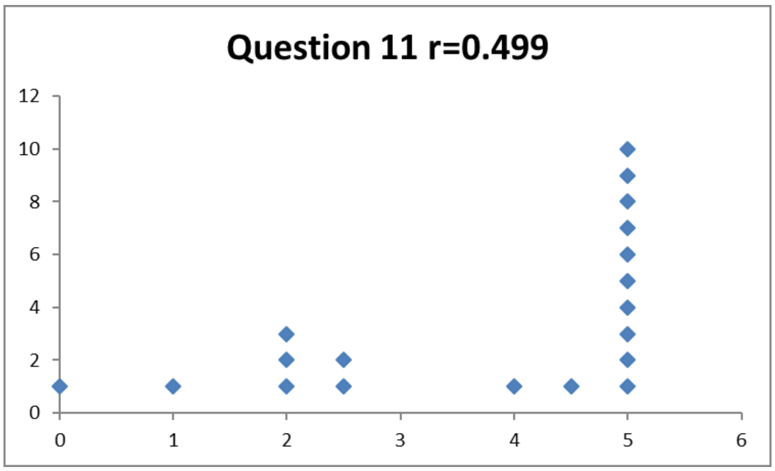
$du = dx$ $\quad v = \tan(x)$



11) Find the integral below. (5 points)

$$\int \sin(x)\cos^4(x)\,dx = -\int u^4 du = -\frac{u^5}{5} + C = -\frac{\cos^5(x)}{5} + C$$

$$u = \cos(x)$$
$$du = -\sin(x)\,dx$$



Part 5: Small Bonus

12) A restaurant offers 20 different entries. Your party has 7 people, and you all want different meals so you can share. How many meal choices are there? Use correct mathematical notation. (2 points)

$$\binom{20}{7} = \frac{20!}{7!\,13!}$$