Name ______Solutions____ Discrete I, Quiz 12

Choose and prove ONE of the following using induction.

1) Let $a_1, a_2, \dots, a_n$ be positive real numbers. Show that the arithmetic mean is larger than the geometric mean:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} > (a_1 a_2 \cdot \cdots \cdot a_n)^{\frac{1}{n}}$$

This problem is a bit of a red-herring because if you do the scratch work to figure it out, you'll notice that it's actually quite difficult. However, we can set up practically everything except some algebra in the inductive step.

Base case: Clearly this is true for $n = 1$:

$$\frac{a_1}{1} = (a_1)^1$$

Induction hypothesis: Assume that

$$\frac{a_1 + a_2 + \cdots + a_k}{k} > (a_1 a_2 \cdot \cdots \cdot a_k)^{\frac{1}{k}}$$

for some index k.

Inductive step:

Consider $\frac{a_1+a_2+\cdots+a_{k+1}}{k+1}$, and without loss of generality, rearrange the terms so that a_{k+1} is the largest number in the summation.

$$\begin{aligned}
\frac{a_1 + a_2 + \cdots + a_{k+1}}{k+1} &= \frac{a_1 + a_2 + \cdots + a_k}{k+1} + \frac{a_{k+1}}{k+1} \\
&> \frac{a_1 + a_2 + \cdots + a_k}{k} + \frac{a_{k+1}}{k+1} \\
&> (a_1 a_2 \cdot \cdots \cdot a_k)^{\frac{1}{k}} + \frac{a_{k+1}}{k+1} \\
&\geq ??? \\
&\geq (a_1 a_2 \cdot \cdots \cdot a_{k+1})^{\frac{1}{k+1}}
\end{aligned}$$

Therefore

$$\frac{a_1 + a_2 + \cdots + a_{k+1}}{k+1} > (a_1 a_2 \cdot \cdots \cdot a_{k+1})^{\frac{1}{k+1}}$$

Hence for all $n = 1, 2, \dots$ we get that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} > (a_1 a_2 \cdot \cdots \cdot a_n)^{\frac{1}{n}}$$

2) Suppose that among n points, any three of them are contained in a circle of radius 1. Show that all n points are contained in a single circle of radius 1. n is at least 3.

Note the underlying assumption: throughout this proof it is assumed that among any three points we can draw a circle of radius 1 containing all three.

Base case: When $n = 3$, it is given that all three points are in one circle of radius 1.

Induction hypothesis: Assume for k points, that a single circle of radius 1 can be found that contains them all.

Inductive step: Consider $k + 1$ points. Create a circle of radius 1 at each point. By a previous theorem, all $k + 1$ of these circles intersect. Use that intersection point as the center of the new circle, which then contains all $k + 1$ points.

Therefore by induction n points will all be contained in a single circle of radius 1, for any index $n \geq 3$.

3) Suppose there are n lines in a plane, no two of which are parallel and no three of which have a common point. Show that the plane is divided into $\frac{n^2+n+2}{2}$ regions.

This is probably the easiest of the three problems, because it doesn't require any outside knowledge.

Base case: One line divides the plane into regions: $\frac{1^2+1+2}{2} = 2$.

Induction hypothesis: Assume that k lines divide the plane into $\frac{k^2+k+2}{2}$ regions.

Induction step: Draw a new line. Because it is not parallel to any lines, it intersects every single previous line. At each intersection an old region is broken into two new regions. Further, after the last intersection the external region on that side is also broken into two new regions. Thus this new line has formed $k+1$ new regions.

$$\frac{k^2+k+2}{2} + k + 1 = \frac{k^2+k+2+2k+2}{2} = \frac{(k^2+2k+1)+(k+1)+2}{2} = \frac{(k+1)^2+(k+1)+2}{2}$$

Therefore by induction, n such lines divide the plane into $\frac{n^2+n+2}{2}$ regions.