Name ___Solutions____ Discrete I, Quiz 14

Choose ONE of the following problems.

1) Define the sequence of real numbers $c_1, c_2, c_3, \ldots$ as follows.

$$c_1 = 0$$
$$c_n = c_{\lfloor \frac{n}{2} \rfloor} + n^2$$

Show that $c_n < 4n^2$ for all indices $n = 1, 2, 3, \ldots$

Base case:

$$c_1 = 0 < 4 \cdot 1^2$$

Is this enough? c_2 depends on c_1, then c_3 depends on c_1 again. c_4 depends on c_2 and so forth, so indeed we see that they'll all be covered.

Induction hypothesis: For some index k, assume each of the following:

$$c_1 < 4 \cdot 1^2$$
$$c_2 < 4 \cdot 2^2$$
$$\vdots$$
$$c_k < 4 \cdot k^2$$

Induction step: We now show that the inequality is true in the $k + 1^{th}$ case:

$$\begin{aligned}
c_{k+1} &= c_{\lfloor \frac{k+1}{2} \rfloor} + (k+1)^2 \\
&< 4 \cdot \left(\left\lfloor \frac{k+1}{2} \right\rfloor \right)^2 + (k+1)^2 \\
&\leq 4 \cdot \left(\frac{k+1}{2} \right)^2 + (k+1)^2 \\
&= 4 \cdot \frac{(k+1)^2}{4} + (k+1)^2 \\
&= (k+1)^2 + (k+1)^2 \\
&= 2(k+1)^2 \\
&< 4(k+1)^2
\end{aligned}$$

Hence $c_{k+1} < 4(k+1)^2$.

Therefore by induction, for all indices n, $c_n < 4 \cdot n^2$.

2) Show that all integers at least 12 can be written in terms of 3 and 5. That is, there are x and y such that $n = 3x + 5y$.

Scratch work:

$$k + 1 = k - 2 + 3$$

Here we see that we'll need the $k - 2^{th}$ case the prove the $k + 1^{th}$ case, so we'll do induction three at a time. This means we'll need three base cases, and three assumptions in the induction hypothesis.

Base cases:

$$12 = 3 \cdot 4$$
$$13 = 2 \cdot 5 + 3$$
$$14 = 3 \cdot 3 + 5$$

Induction hypothesis: Assume each of the following for some index k:

$$k = 3a + 5b$$
$$k - 1 = 3c + 5d$$
$$k - 2 = 3e + 5f$$

Induction step: We now prove the $k + 1^{th}$ case:

$$\begin{aligned} k + 1 &= k - 2 + 3 \\ &= 3e + 5f + 3 \\ &= 3e + 3 + 5f \\ &= 3(e + 1) + 5 \end{aligned}$$

Therefore $k + 1$ can be written in terms of $3s$ and $5s$. Hence by induction all integers at least 12 can be written in this form.