Name ___Solutions____ Discrete I, Test 2, 10/24/2014

1) Show that $3n + 1 < 3^n$ for all $n = 6, 7, 8, \ldots$ (20 points)

Notes regarding common mistakes:

- Make sure in your induction hypothesis that you're assuming the statement itself. The sentence should be phrased such that the assumption is the statement "$3k + 1 < 3^k$".
- In the induction step do not assume your conclusion!! We must prove that $3(k+1) + 1 < 3^{k+1}$. This requires proof! Starting out by stating what you want to prove is circular reasoning.
- Did the proctor let you use calculators?? Some people seemed to know fairly obscure numbers such as 3^6.

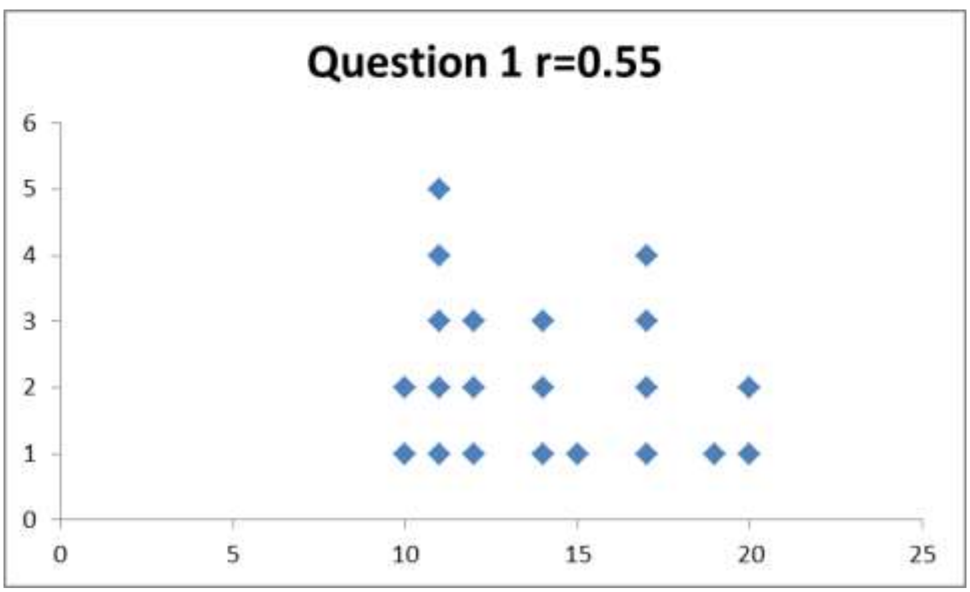
Base case: $3 \cdot 6 + 1 = 19 < 3^6$

Induction hypothesis: Assume that $3k + 1 < 3^k$ for some index k.

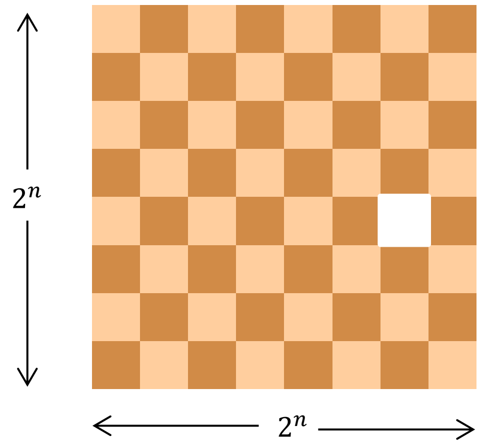
Induction step:

$$3(k+1) + 1 = 3k + 3 + 1 = 3k + 1 + 3 < 3^k + 3 < 3^k + 3^k = 3^{k+1}$$

Therefore by induction $3n + 1 < 3^n$ for all indices n.



2) Show that a $2^n \times 2^n$ chess board with one random tile removed can be tiled with the piece shown below. (20 points)

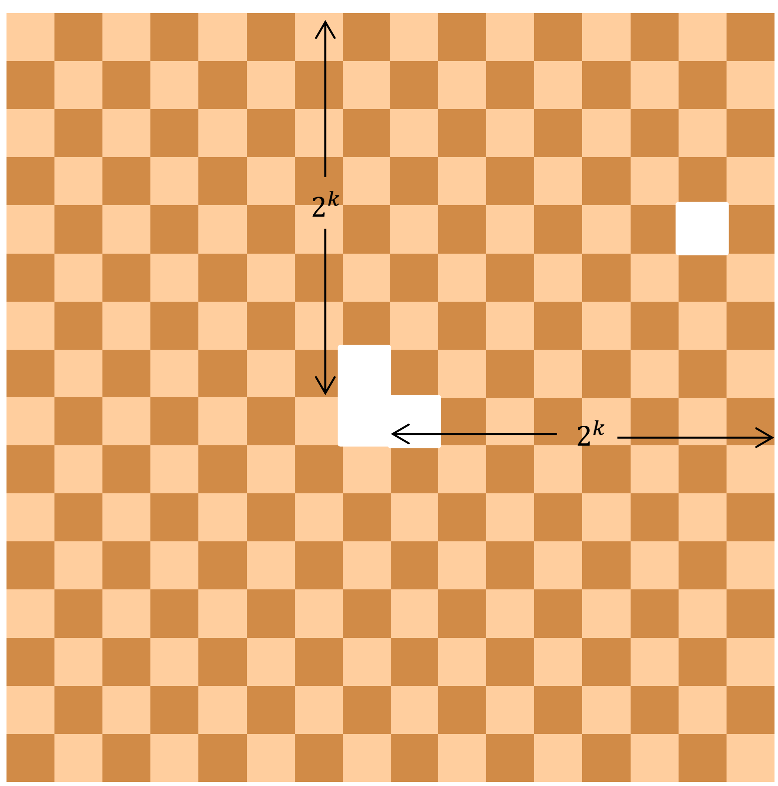


I goofed on this question: on two versions the statement was false. On those versions this question was thus not counted; but if you did realize that it was false, a few extra credit points were awarded.

Base case: The smallest index that makes sense is $n = 1$. Indeed in this case we can tile a 2×2 deficient board by rotating the piece as shown below.

Induction hypothesis: Assume that all $2^k \times 2^k$ deficient boards can be tiled.

Inductive step: Consider a $2^{k+1} \times 2^{k+1}$ board. Decompose it into four $2^k \times 2^k$ deficient boards as shown below. Each of these can be tiled by the induction hypothesis. Hence by induction all $2^n \times 2^n$ deficient boards can be tiled.



3) Show that all integers $n \geq 15$ can be written as:

$$n = 3x + 7y$$

where x and y are integers. (20 points)

The typical approach this problem is to use three base cases, three induction hypotheses, and an inductive step that relies on one of the three previous cases.

Base cases:

$$15 = 3 \cdot 5$$
$$16 = 3 \cdot 3 + 7$$
$$17 = 3 + 7 \cdot 2$$

Induction hypotheses: For some index k, assume each of the following:

$$k = 3a + 7b$$
$$k - 1 = 3c + 7d$$
$$k - 2 = 3e + 7f$$

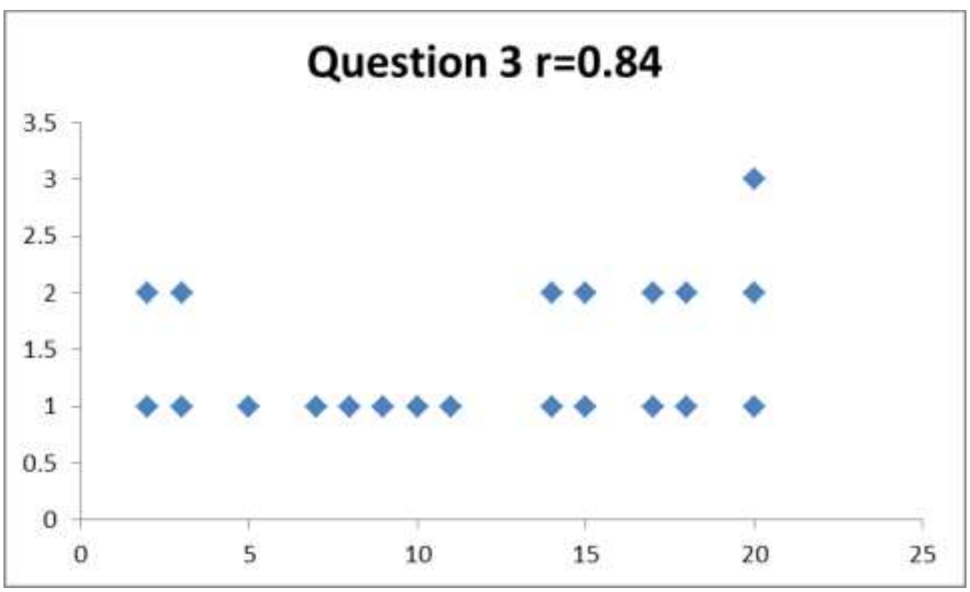
where a, b, c, d, e and f are integers.

Inductive step:

$$k + 1 = k - 2 + 3 = 3e + 7f + 3 = 3(e + 1) + 7f$$

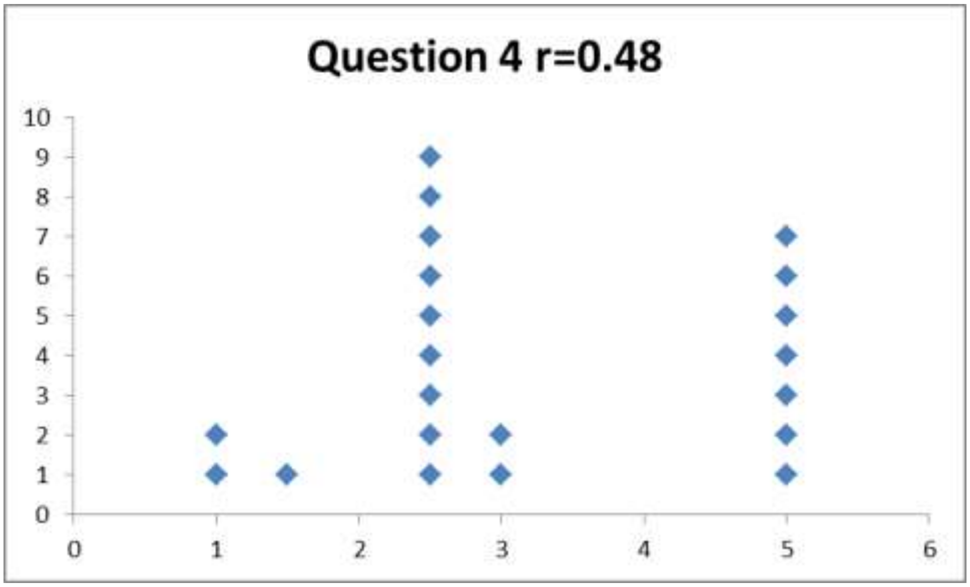
Hence $k + 1$ can be written in terms of 3 and 7.

Therefore by induction all integers at least 15 can be written in terms of 3 and 7.



4) Let $f(x) = [3x]$ with domain $\mathbb{R}$. What is the range of f? (5 points)

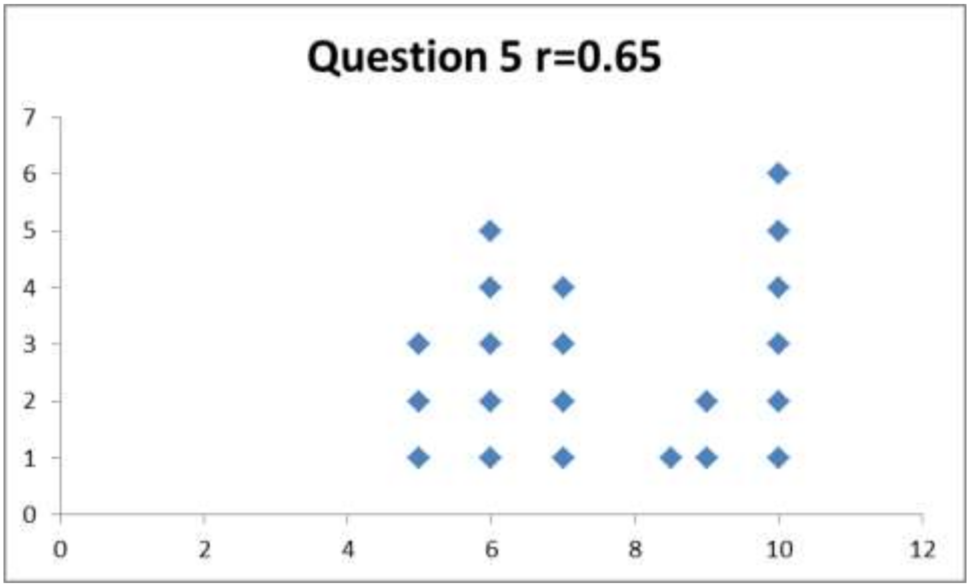
The range is the set of all integers: $\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$



5) Is the function $f(x) = 3x + 1$ one to one? Justify your answer. (10 points)

Indeed this function is one to one. We see below that every output comes from just one input:

$$f(x_2) = f(x_1)$$
$$\therefore 3x_2 + 1 = 3x_1 + 1$$
$$\therefore 3x_2 = 3x_1$$
$$\therefore x_2 = x_1$$

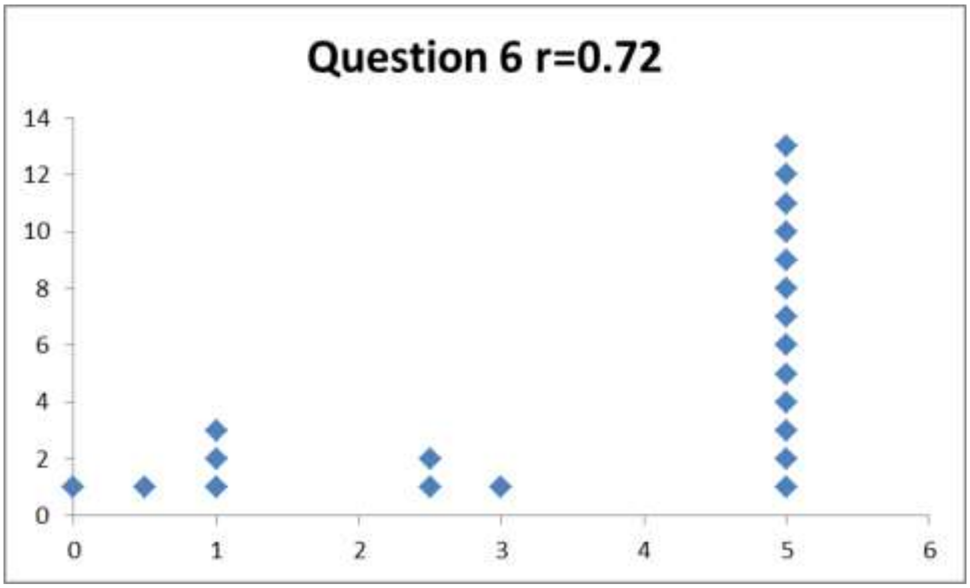


6) Decompose the function $f(x) = (2x + 4)^3$ into two simpler functions. Be sure to explicitly give the rule for both of your answers. (5 points)

There are multiple answers. The easiest is probably:

$$g(x) = x^3$$
$$h(x) = 2x + 4$$

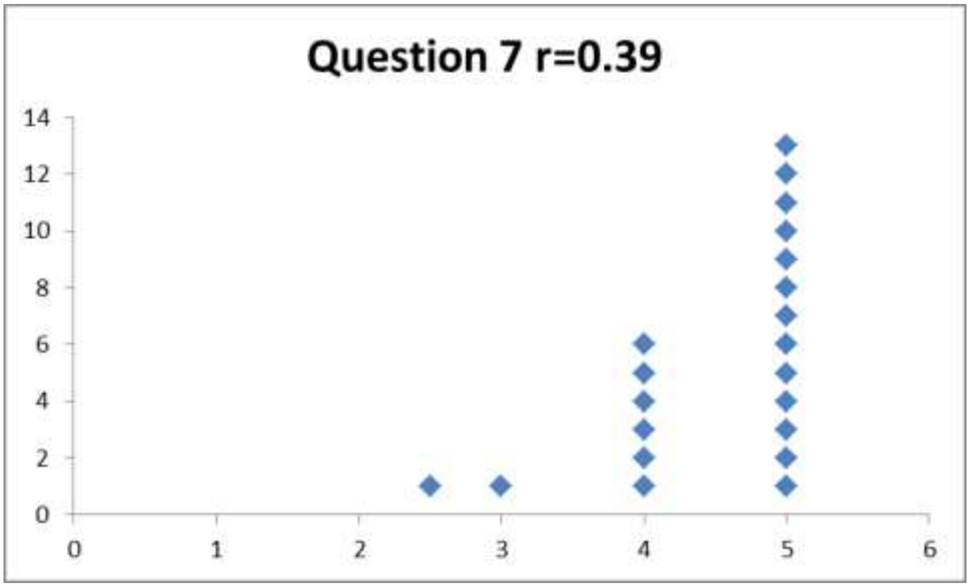
Then we have $f = g \circ h$.



7) Reindex the summation below to start at $i = 0$. (5 points)

$$\sum_{i=3}^{n+17} 2^i$$

$$\sum_{i=0}^{n+14} 2^{i+3}$$



8) Give an example of a sequence that satisfies each of the following: (5 points)

Nondecreasing

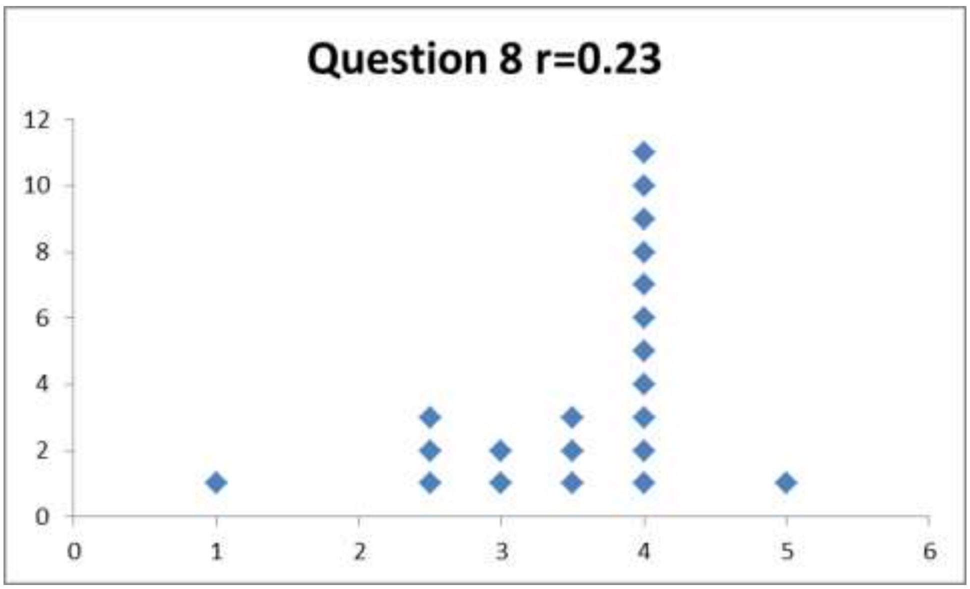
Not increasing

Not constant

Give an example of a sequence… not three different sequences. So many people made this mistake that I only deduced one point for it.

There are many answers, one such answer is:

$$5, 5, 6, 6, 7, 7, 8, 8, 9, 9, \ldots$$



9) List all strings of length at most 2 that are members of $\{a, b\}^*$. (5 points)

"aa"

"ab"

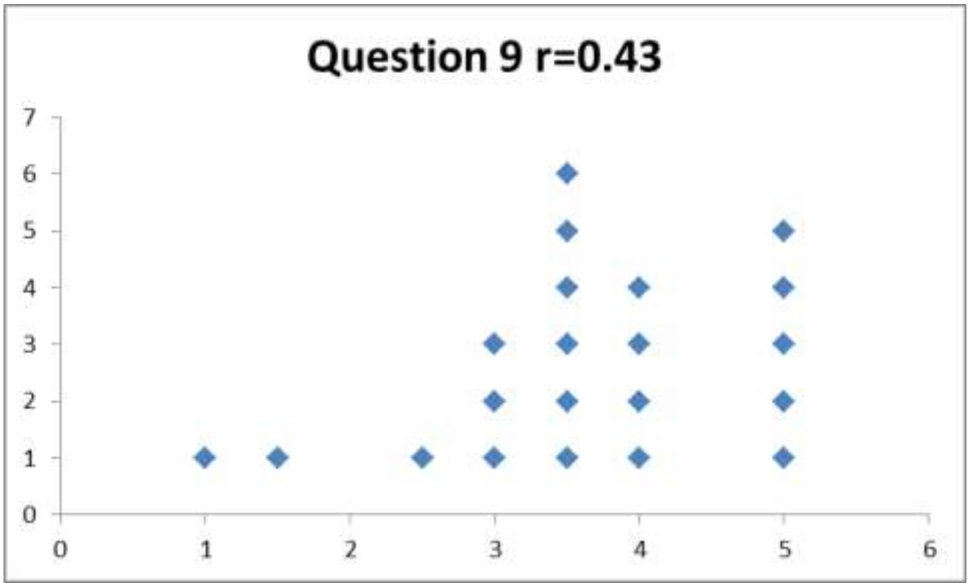
"ba"

"bb"

"a"

"b"

""



10) List all the substrings of the string "`abc`". (5 points)

"abc"

"ab"

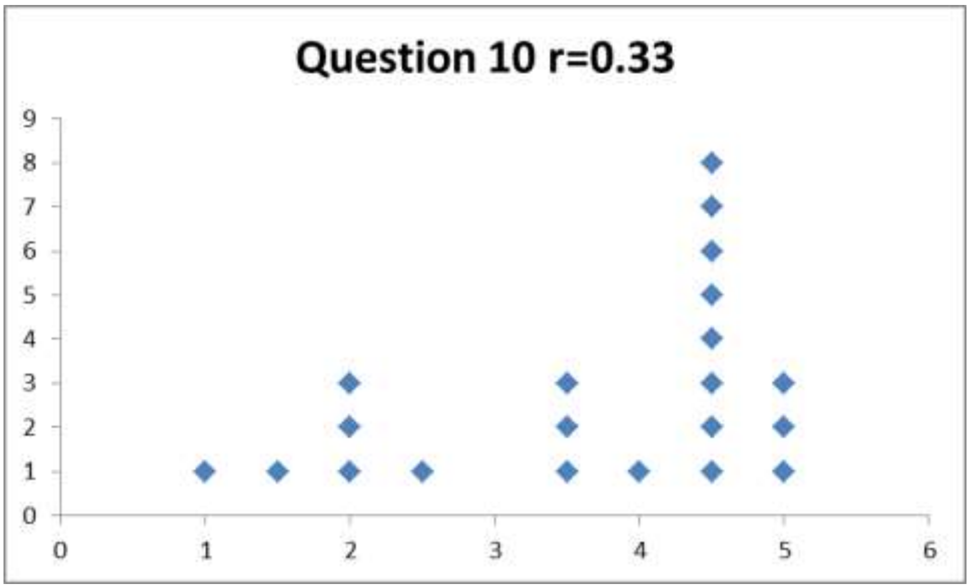
"bc"

"a"

"b"

"c"

""



11) Let H_n be the n^{th} harmonic number as defined below.

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Show the following for all $m = 1, 2, 3, \ldots$

$$\sum_{n=1}^{m} H_n = (n+1)H_n - n$$

(20 bonus points)

I goofed on this problem too. The RHS should involve $m's$, not $n's$:

$$\sum_{n=1}^{m} H_n = (m+1)H_m - m$$

It was still quite possible to make a clear and meaningful start on this problem. Some bonus points were awarded in such a case.