Name __Solutions____ Discrete I, Quiz 5

1) Identify which of these functions are one-to-one and onto by CIRCLING the functions that are one-to-one and BOXING the ones that are onto.

(a) $f_1(x) = 2x + 3$ with domain $\mathbb{R}$ and codomain $\mathbb{R}$.

(b) $f_2(x) = 2x + 3$ with domain $\mathbb{Z}$ and codomain $\mathbb{Z}$.

(c) $f_3(x) = x^2$ with domain $\mathbb{R}$ and codomain $\mathbb{R}$.

(d) $f_4(x) = x^2$ with domain $\mathbb{Z}$ and codomain $\mathbb{Z}$.

(e) $f_5(x) = \lfloor x \rfloor$ with domain $\mathbb{R}$ and codomain $\mathbb{Z}$.

2) Define the numbers $c_0, c_1, c_2, c_3, c_4, \ldots$ via $c_0 = 1$ and $c_n = c_{\lfloor \frac{n}{3} \rfloor} + \frac{4}{3}$. Prove, using strong induction, that $c_n < 2n$ for all $n \geq 1$.

On this problem I goofed – it's only true for $n \geq 2$! So what do you do when you come up against something that is not true? The best thing to do is to ask the professor. The second best thing is to write a note and explain why you think it's not true.

As such, this problem was graded as follows:
If you didn't notice the problem: Graded as normal.
If you noticed the problem, explained it, and stopped there: automatic full credit (5 points).
If you noticed the problem, but humored it anyway to show that you know how to do induction: Graded out of 10 points instead of 5.

Base Case: After noticing that $c_1 = c_0 + \frac{4}{3} = \frac{7}{3} \not< 2 \cdot 1$, we throw that case out. However, it does for for $n = 2$ becase:

$$c_2 = c_{\lfloor \frac{2}{3} \rfloor} + \frac{4}{3} = c_0 + \frac{4}{3} = 1 + \frac{4}{3} = \frac{7}{3} < 4 = 2 \cdot 2$$

**One must also check a couple more cases, until the induction step starts to apply.

Induction Hypothesis: Let k be an arbitrary integer and assume each of the following inequalities hold:

$$\begin{gathered} c_2 < 2 \cdot 2 \\ c_3 < 2 \cdot 3 \\ \vdots \\ c_k < 2 \cdot k \end{gathered}$$

Induction Step: Now we prove the inequality is true when $n = k + 1$.

$$c_{k+1} = c_{\lfloor \frac{k+1}{3} \rfloor} + \frac{4}{3} < 2 \cdot \left\lfloor \frac{k+1}{3} \right\rfloor + \frac{4}{3} \leq 2\left(\frac{k+1}{3}\right) + \frac{4}{3} = \frac{2k}{3} + \frac{2}{3} + \frac{4}{3} = \frac{2k}{3} + 2 < 2k + 2 = 2(k+1)$$