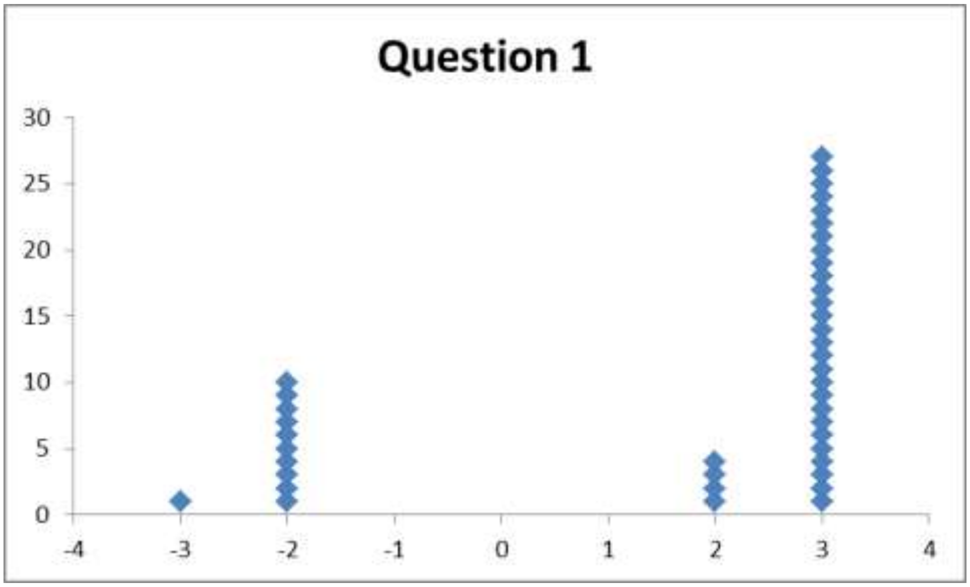
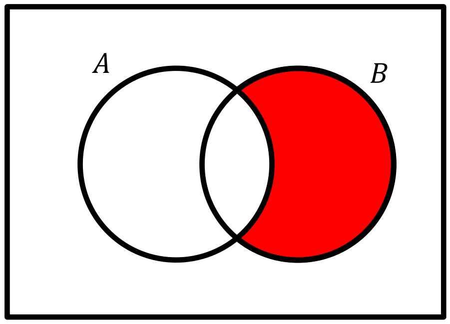
Name __Solutions____ Discrete I, Test 1, Fall 2015

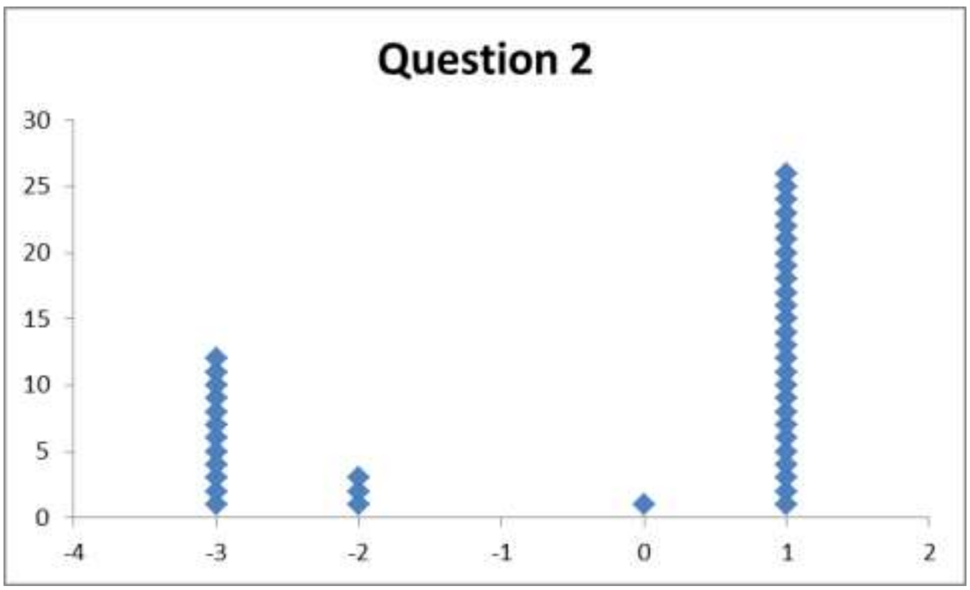
1) Let $A = \{1,2,3,4\}, B = \{a, b, c, d\}, C = \{1,2, a, b, 5,6\}$. Find $(C \cap B) - A$. (-3/+3 points)

$\{a, b\}$



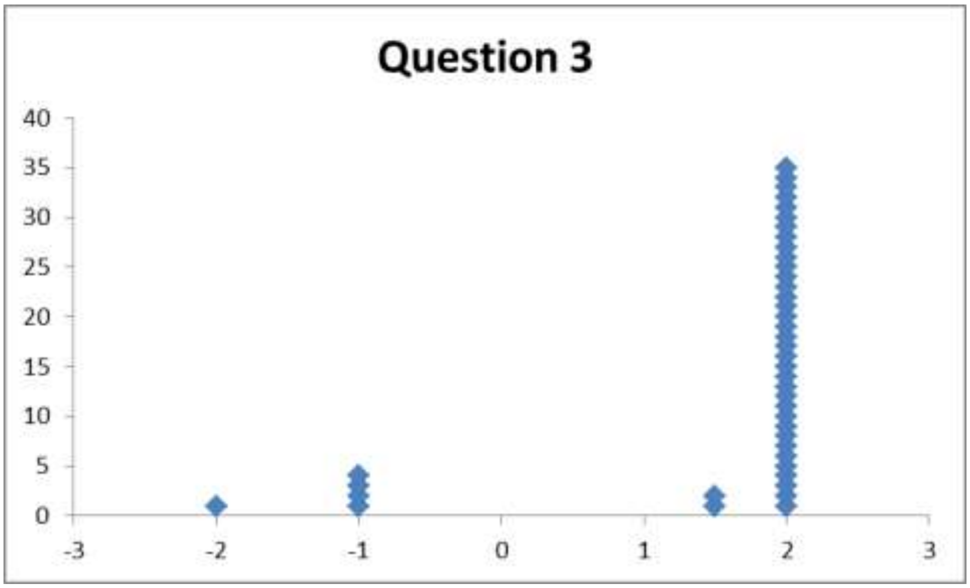
2) On the Venn Diagram below, shade the set $A^c \cap B$. (-3/+1 points)





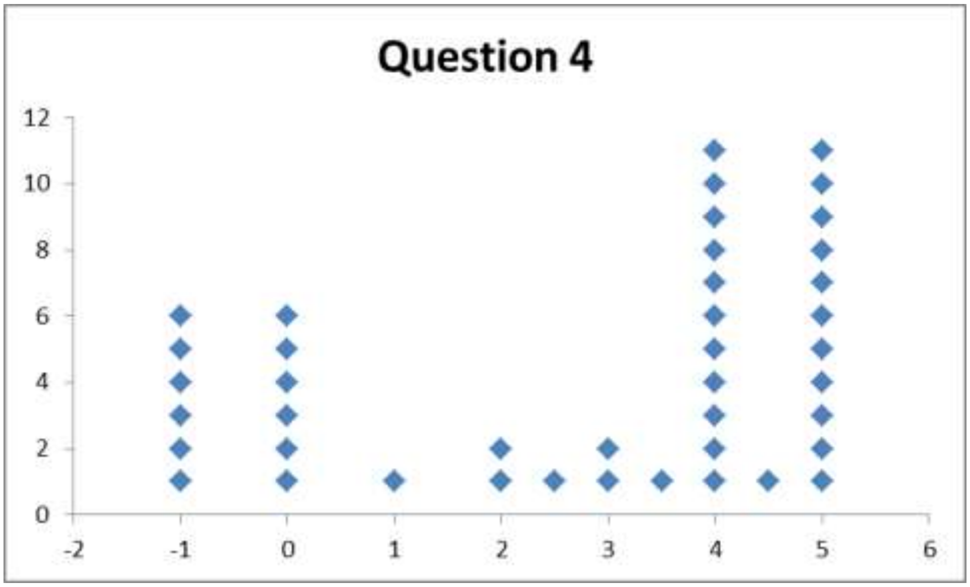
3) Write a truth table for the statement $(p \vee q) \wedge \neg p$ (-2/+2 points)

p	q	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p$
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

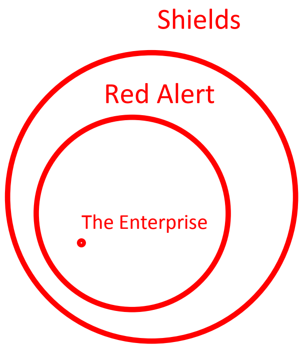


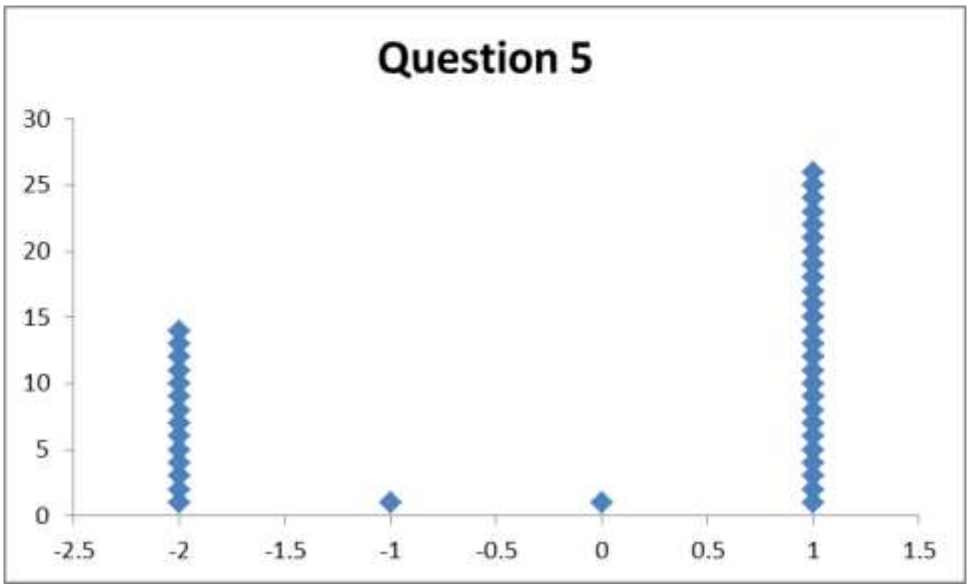
4) Suppose you have 11 dollars and 5 friends. You wish to give all the money to your friends. Justify the claim that at least one friend gets more than 2 dollars. (-1/+5 points)

If you give each friend the most you could - $2 each, then you've only given away $10. Whoever gets the extra money will have more than $2.



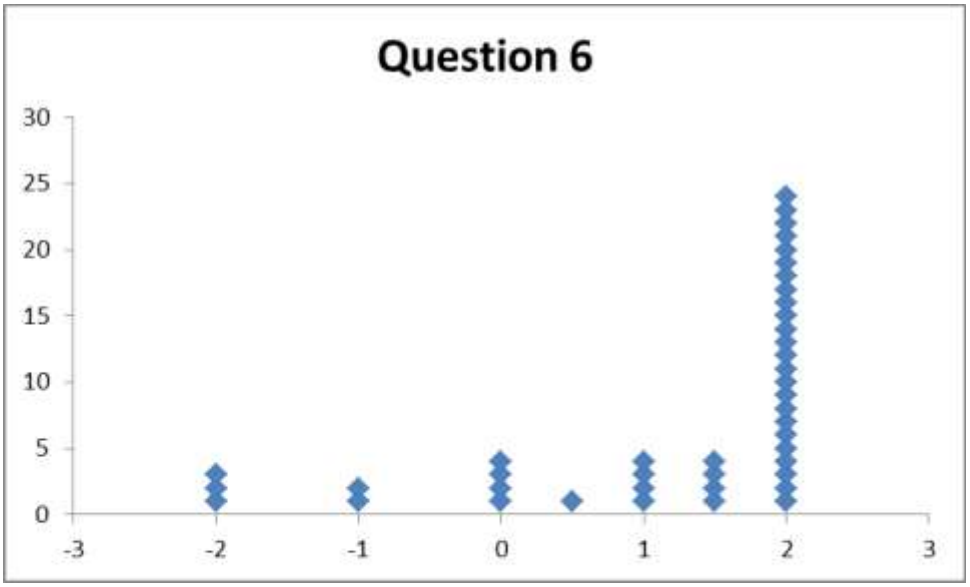
5) Use a Venn Diagram to illustrate the statement “When Red Alert is activated, the shields go up”. (-3/+1 points)





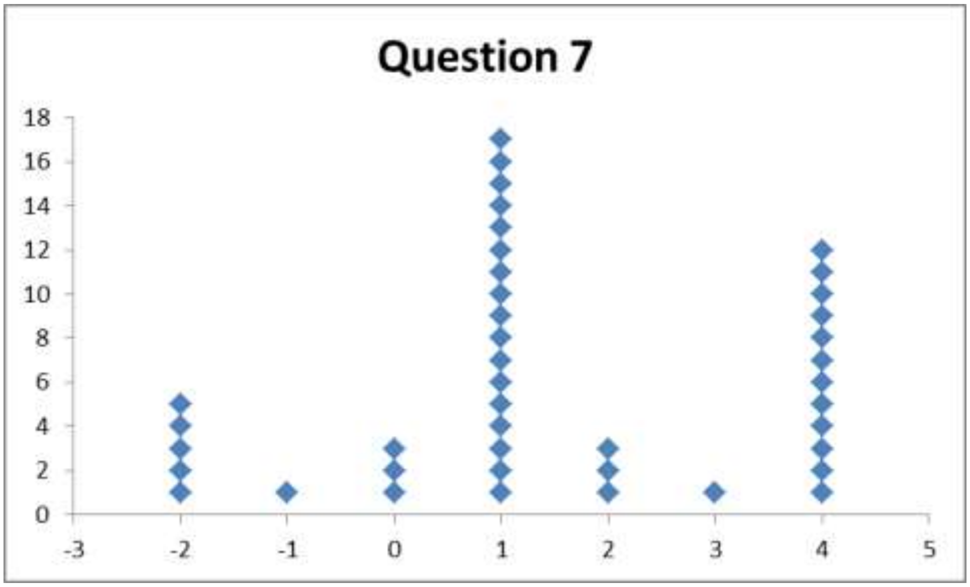
6) Determine whether or not the argument below is valid. Illustrate this on your Venn Diagram above. "When Red Alert is activated, the shields go up. The Enterprise is under Red Alert. Therefore the shields are up." (-2/+2 points)

This is valid, as can be seen from the mark for the Enterprise above: it is within the set for shields.



7) Write a complete and grammatically correct sentence that has the form "$\forall_x\big(P(x)\big)$" (-2/+4 points)

Everybody will get this question right.



Determine whether each of the following is true or false. (-0.5/+0.5 points each)

8) $\forall_{x\in\mathbb{R}}(x^3 \geq 0)$ False

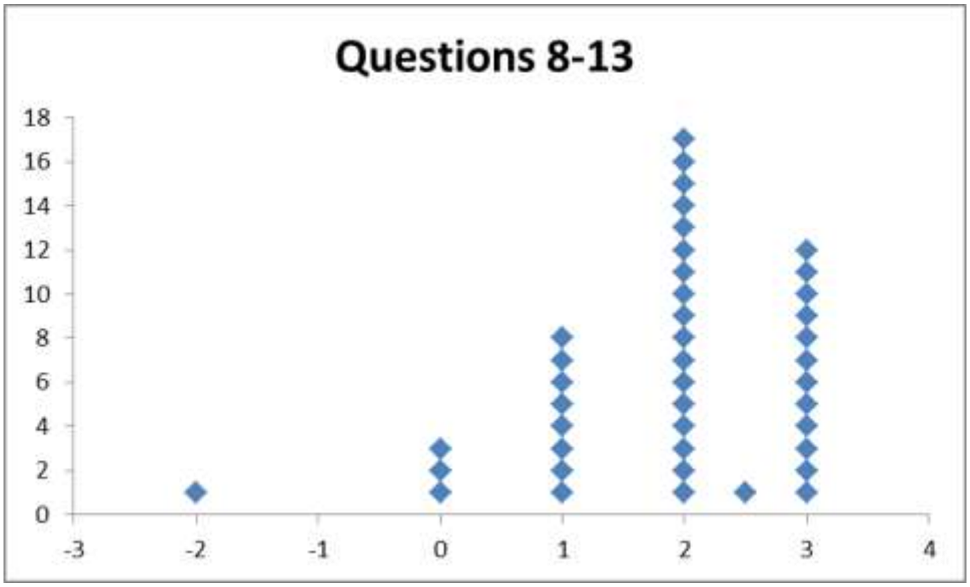
9) $\exists_{x\in\mathbb{R}}(x^3 \geq 0)$ True

10) $\forall_{x\in\mathbb{R}}\exists_{y\in\mathbb{R}}(xy = 1)$ True

11) $\exists_{x\in\mathbb{R}}\forall_{y\in\mathbb{R}}(xy = 1)$ False

12) $\forall_{x\in\mathbb{R}}\forall_{y\in\mathbb{R}}(xy = 1)$ False

13) $\exists_{x\in\mathbb{R}}\exists_{y\in\mathbb{R}}(xy = 1)$ True



14) For all rational numbers x and y, prove that $x - y$ is rational. (-2/+4 points)

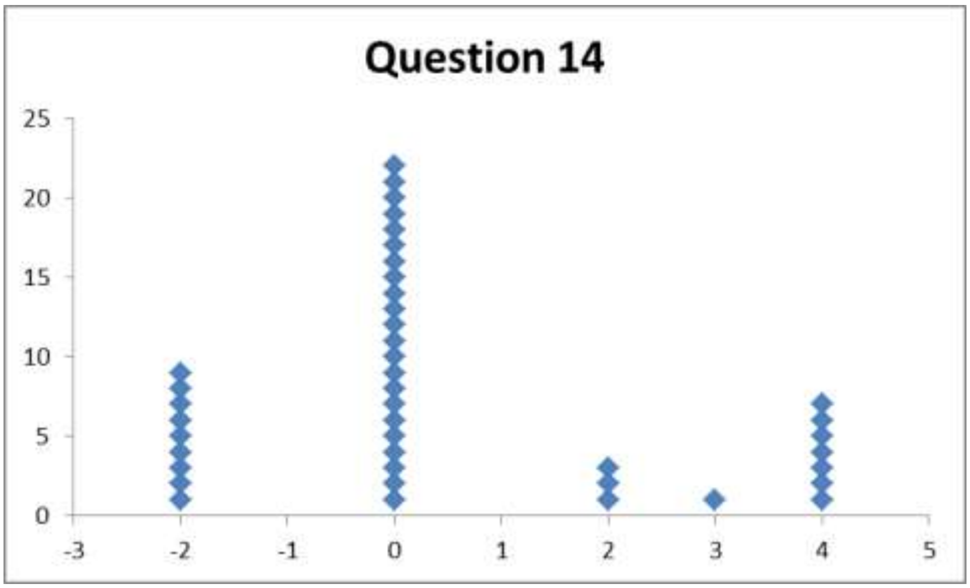
A rational number is one that can be written as a fraction of integers. Thus we will write:

$$x = \frac{a}{b} \text{ and } y = \frac{c}{d}$$

Now we find and simplify $x - y$:

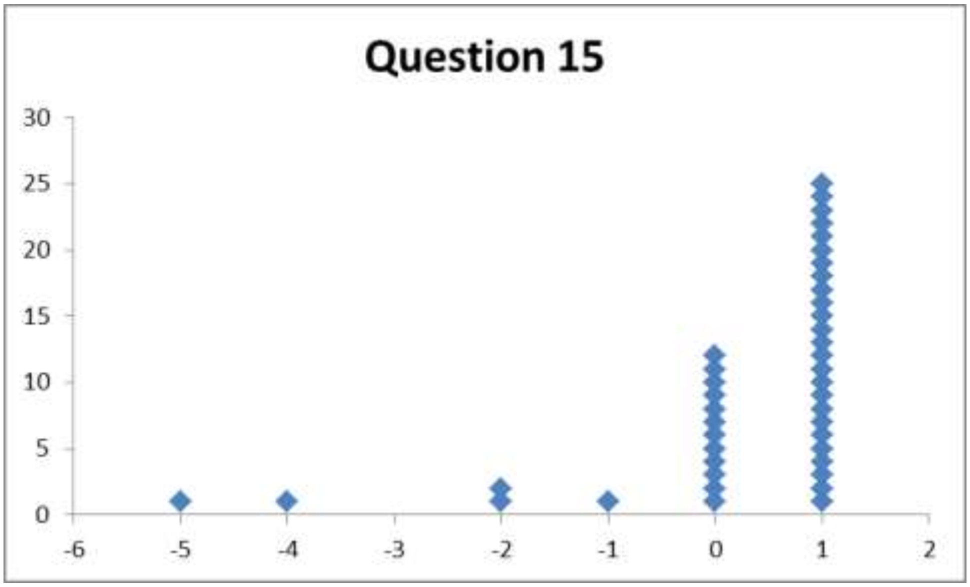
$$x - y = \frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{cb}{bd} = \frac{ad - bc}{bd}$$

Both the numerator and denominator are integers, and so indeed $x - y$ is rational!



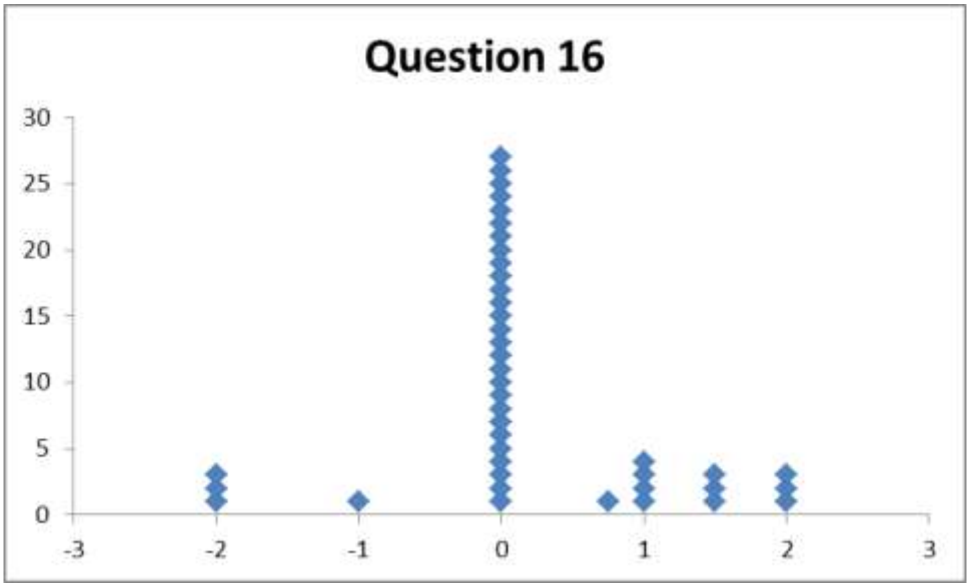
15) Prove that there exist integers x, y and z such that $xy = z^2$. (-5/+1 points)

Choose $x = y = z = 0$, then we have $xy = 0 = z^2$.



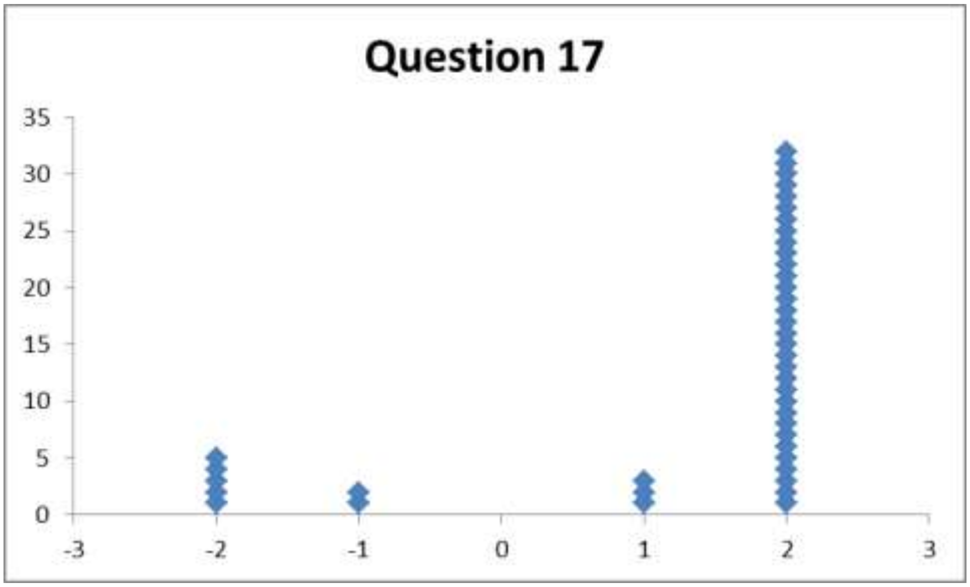
16) Find $\mathcal{P}(\{a,b,c\})$. (-2/+2 points)

$$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$



17) Write a complete and grammatically correct sentence that has the form "$p \vee q$". (-2/+2 points)

Either you answered this question correctly, or you answered it incorrectly.



18) Prove the equality below for all integers $n \geq 1$. (-10/+10 points)

$$\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$$

BC:

$$\sum_{i=1}^{1} \frac{1}{(2i-1)(2i+1)} = \frac{1}{(2-1)(2+1)} = \frac{1}{3} = \frac{1}{2+1}$$

IH: For some integer positive k, assume:

$$\sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)} = \frac{k}{2k+1}$$

IS:

Write out the summation

$$\sum_{i=1}^{k+1} \frac{1}{(2i-1)(2i+1)} = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$= \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)}$$ Simplify the last term

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$ Apply the IH

$$= \frac{k(2k+3)}{(2k+1)(2k+3)} + \frac{1}{(2k+1)(2k+3)}$$ Common Denominator

$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$ Combine Fractions

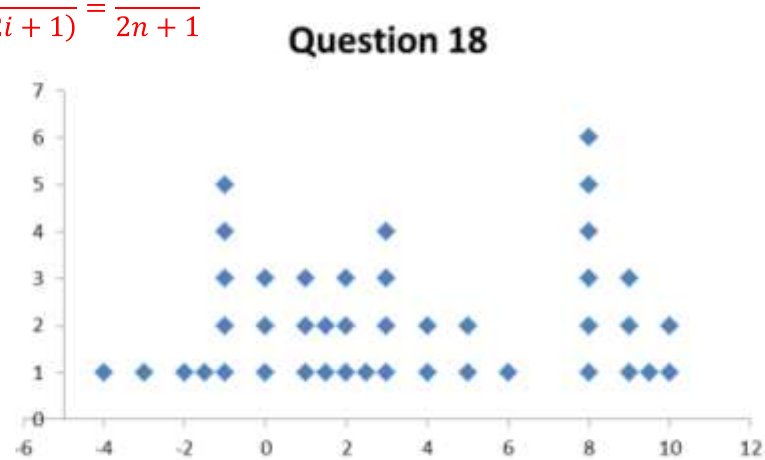
$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$ Expand

$$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$ Factor

$$= \frac{(k+1)}{(2k+3)}$$ Cancel $2k+1$

Therefore for all n,

$$\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$$



Point distribution:
BC: -1/+1 point
IH: -3/+3 points
IS: -6/+6 points

19) Show that every integer at least 6 can be written as a sum of 2s and 7s. That is, for every $n \geq 6$ there are integers x and y such that $n = 2x + 7y$. (-10/+10 points)

This is one of those problems that requires two base cases because the "next" next in the induction step requires a jump of two.

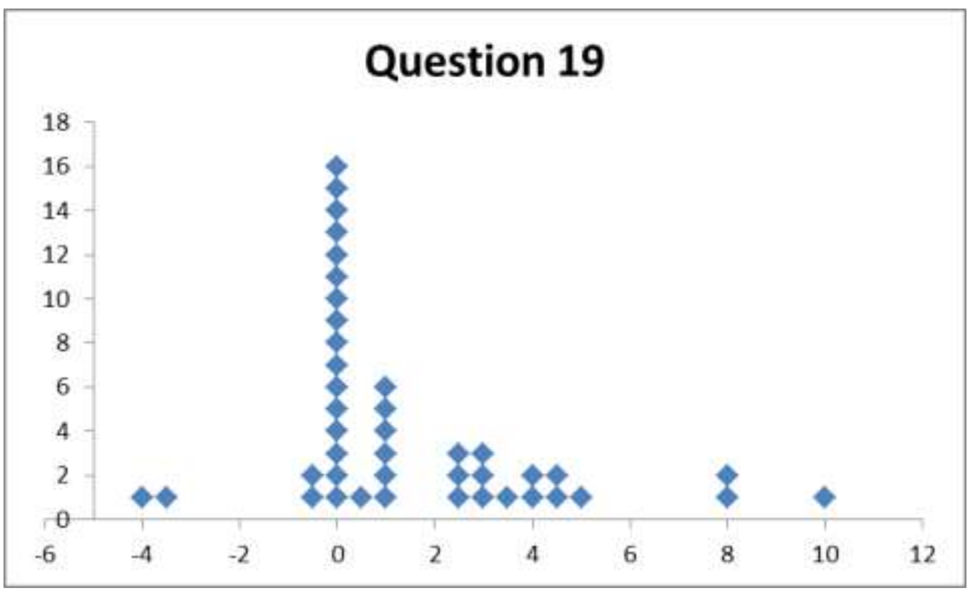
BCs: $6 = 2 \cdot 3 + 7 \cdot 0$; $7 = 2 \cdot 0 + 7$.

IHs: Assume that there are integers x_1, x_2, y_1, y_2 such that $k = 2x_1 + 7y_1$ and $k + 1 = 2x_2 + 7y_2$.

IS: We will now show that there are integers x_3 and y_3 such that $k + 2 = 2x_2 + 7y_2$.

$$k + 2 = 2x_1 + 7y_1 + 2 = 2x_1 + 2 + 7y_1 = 2(x_1 + 1) + 7y_1$$

Thus we take $x_3 = x_1 + 1$ and $y_3 = y_1$ and we see that indeed $k + 2 = 2x_2 + 7y_2$.



Point distribution:
BCs: -1/+1 point
IHs: -3/+3 points
IS: -6/+6 points

Bonus Question

20) Prove that $\sqrt{7}$ is irrational. (-0/+10 points)