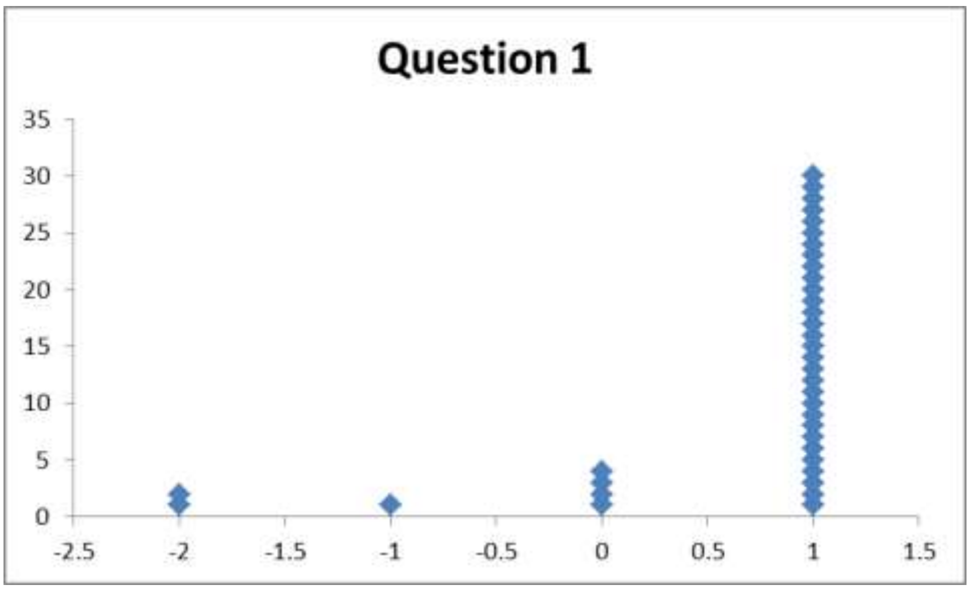
Name ____ Discrete I, Test 2, Fall 2015

For all the problems on this page, use the function defined below. If you need more space, give your answer to the right and mark which problem it completes.

$$f: \{1,2,3\} \to \{0,1,2,3,4,5,6,7,8,9\}$$
$$x \mapsto x^2$$

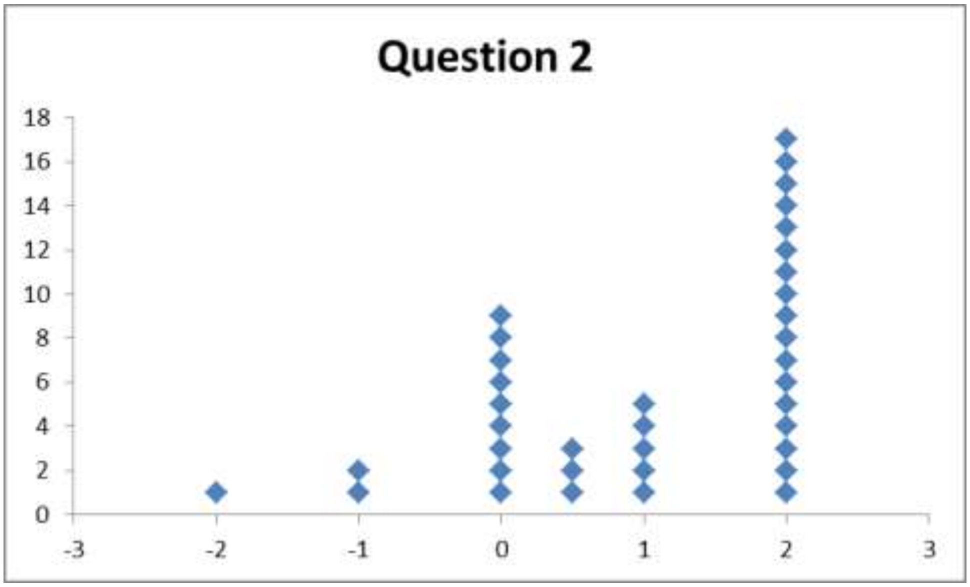
1) What is the domain of f? (-2/+1 points)

$\{1,2,3\}$



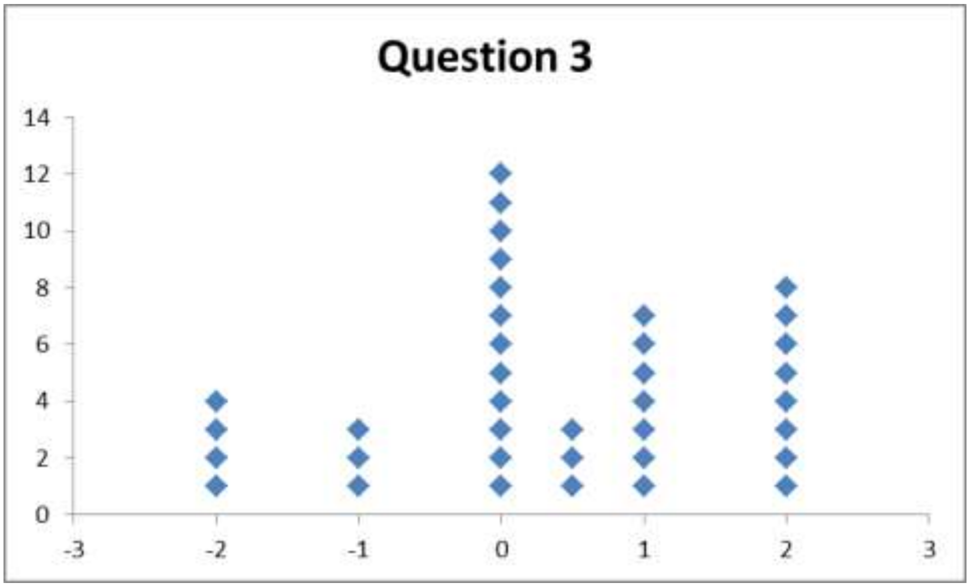
2) What is the codomain of f? (-2/+2 points)

{0,1,2,3,4,5,6,7,8,9}



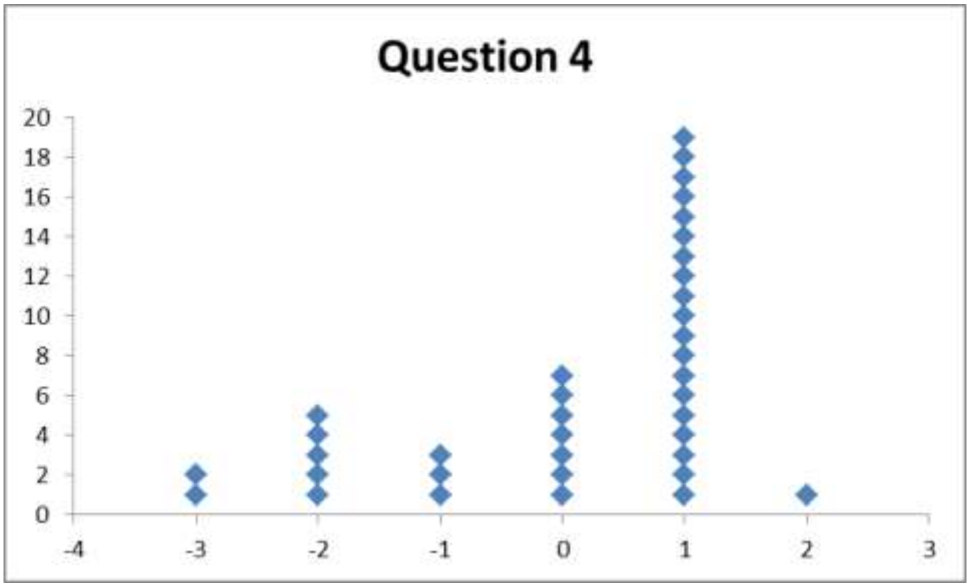
3) What is the range of f? (-2/+2 points)

{1,4,9}



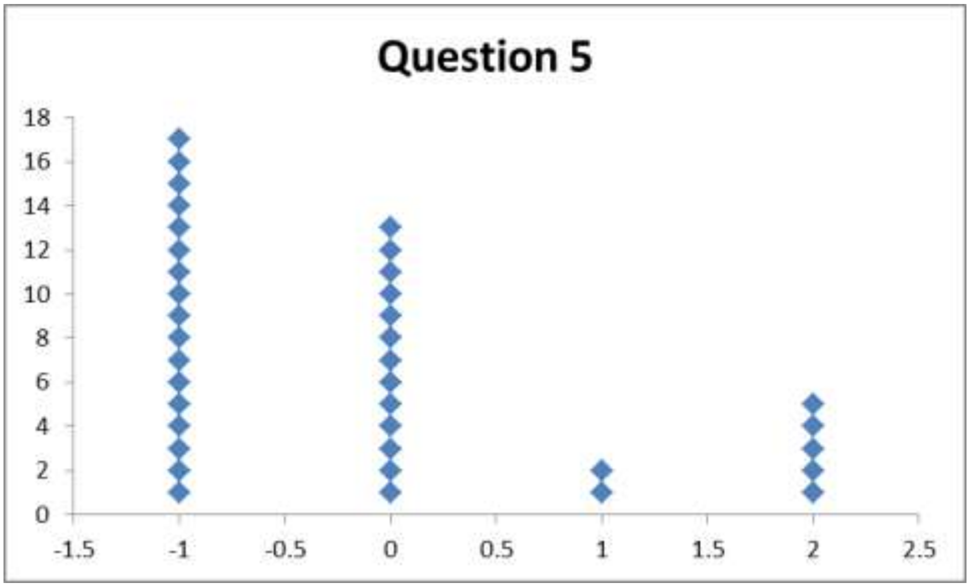
4) What is $f^{-1}(4)$? (-3/+1 points)

2



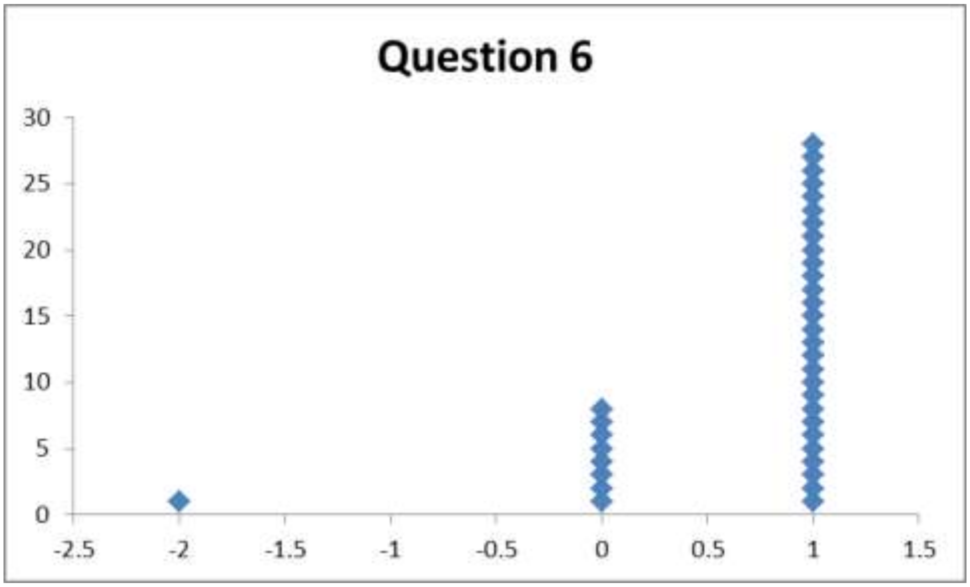
5) What is $f^{-1}(5)$? (-1/+2 points)

This does not exist

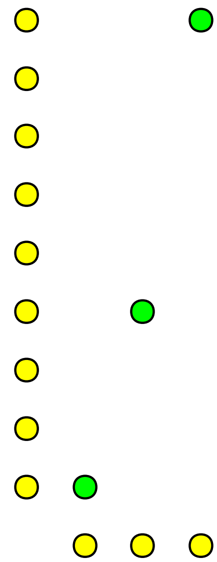


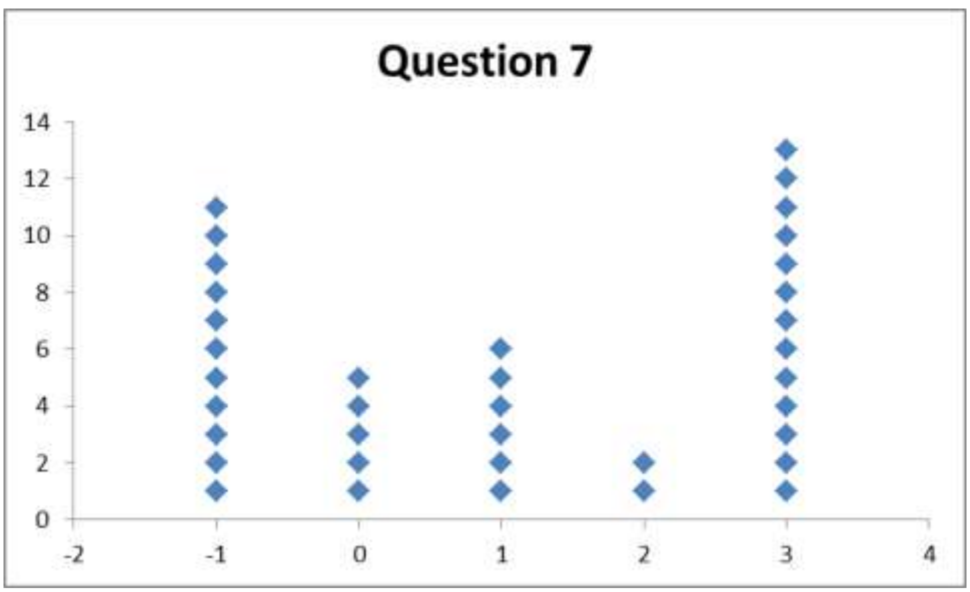
6) What is $f(2)$? (-3/+1 points)

4

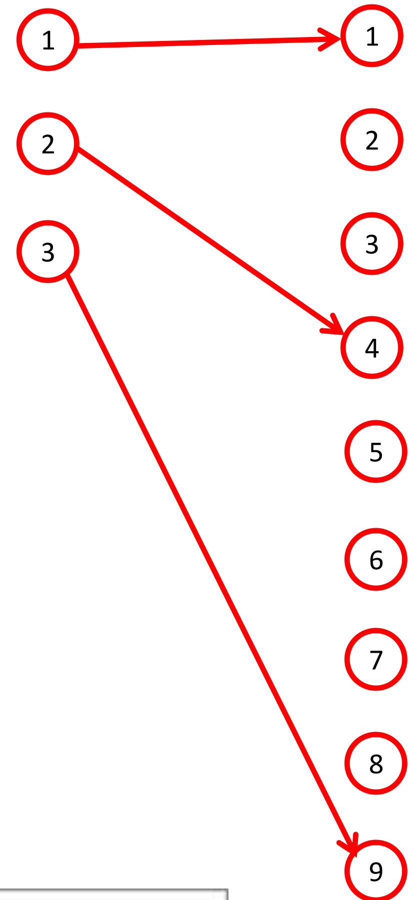


7) Sketch a graph of f. (-1/+3 points)



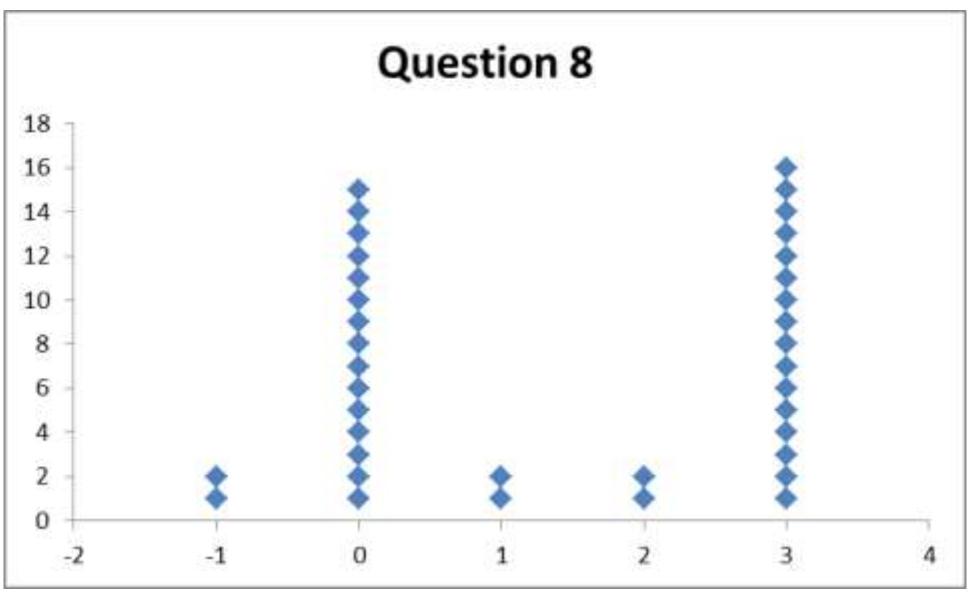


8) Use an arrow diagram to illustrate f. (-1/+3 points)



The diagram to the right is correct.

Credit was given for another interesting answer though. Some people expanded the domain to $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and made the digraph of f, treating it as a relation. Technically this is answering a different question, but it contains essentially the same information.



9) Let S be the range of f, and a new function $g: \{1,2,3\} \to S$ is defined via $g(x) = f(x)$. What is g^{-1}? Define it completely. (bonus question; -0/+5 points)

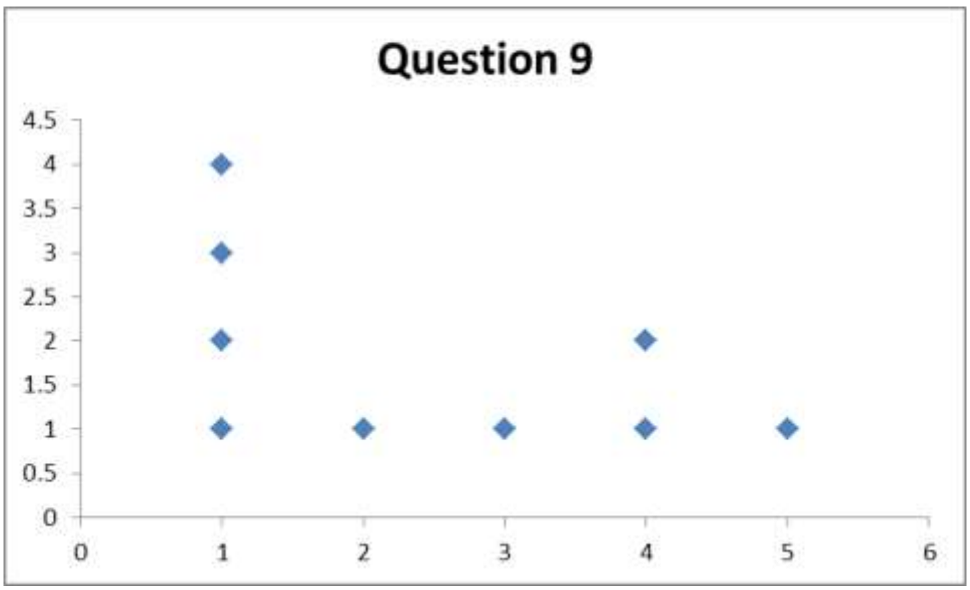
$$g^{-1}: \{1,4,9\} \to \{1,2,3\}$$
$$x \mapsto \sqrt{x}$$

OR

$g: \{1,4,9\} \to \{1,2,3\}$ defined by $g(x) = \sqrt{x}$

OR

$g(x) = \sqrt{x}$ with domain $\{1,4,9\}$ and codomain $\{1,2,3\}$.



10) Define the numbers $c_0, c_1, c_2, c_3, c_4, \ldots$ via $c_0 = 1$ and $c_n = c_{\lfloor \frac{n}{3} \rfloor} + \frac{4}{3}$. Prove, using strong induction, that $c_n < 2n$ for all $n \geq 2$. (-10/+10 points)

Base Case:

$$c_2 = c_{\lfloor \frac{2}{3} \rfloor} + \frac{4}{3} = c_0 + \frac{4}{3} = 1 + \frac{4}{3} = \frac{7}{3} < 4 = 2 \cdot 2$$

Or is this the only base case that is needed? Let's see. c_3 is defined in terms of c_1 which is not covered. Hence we must also show that $c_3 < 2 \cdot 3$:

$$c_3 = c_{\lfloor \frac{3}{3} \rfloor} + \frac{4}{3} = c_1 + \frac{4}{3} = \frac{7}{3} + \frac{4}{3} = \frac{11}{3} < 6 = 2 \cdot 3$$

How about c_4? That is also defined in terms of c_1, as is c_5 so we need to check those as well:

$$c_4 = c_{\lfloor \frac{4}{3} \rfloor} + \frac{4}{3} = c_1 + \frac{4}{3} = \frac{7}{3} + \frac{4}{3} = \frac{11}{3} < 8 = 2 \cdot 4$$
$$c_5 = c_{\lfloor \frac{5}{3} \rfloor} + \frac{4}{3} = c_1 + \frac{4}{3} = \frac{7}{3} + \frac{4}{3} = \frac{11}{3} < 10 = 2 \cdot 5$$

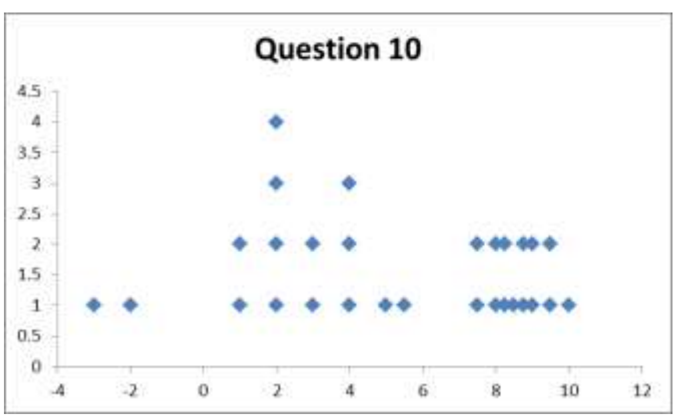
How about c_6? That is defined in terms of c_2, and so finally induction will start to take hold.

Induction Hypothesis: Let k be an arbitrary integer and assume each of the following inequalities hold:

$$c_2 < 2 \cdot 2$$
$$c_3 < 2 \cdot 3$$
$$\vdots$$
$$c_k < 2 \cdot k$$

Induction Step: Now we prove the inequality is true when $n = k + 1$.

$$c_{k+1} = c_{\lfloor \frac{k+1}{3} \rfloor} + \frac{4}{3} < 2 \cdot \left\lfloor \frac{k+1}{3} \right\rfloor + \frac{4}{3} \leq 2\left(\frac{k+1}{3}\right) + \frac{4}{3} = \frac{2k}{3} + \frac{2}{3} + \frac{4}{3} = \frac{2k}{3} + 2 < 2k + 2 = 2(k+1)$$



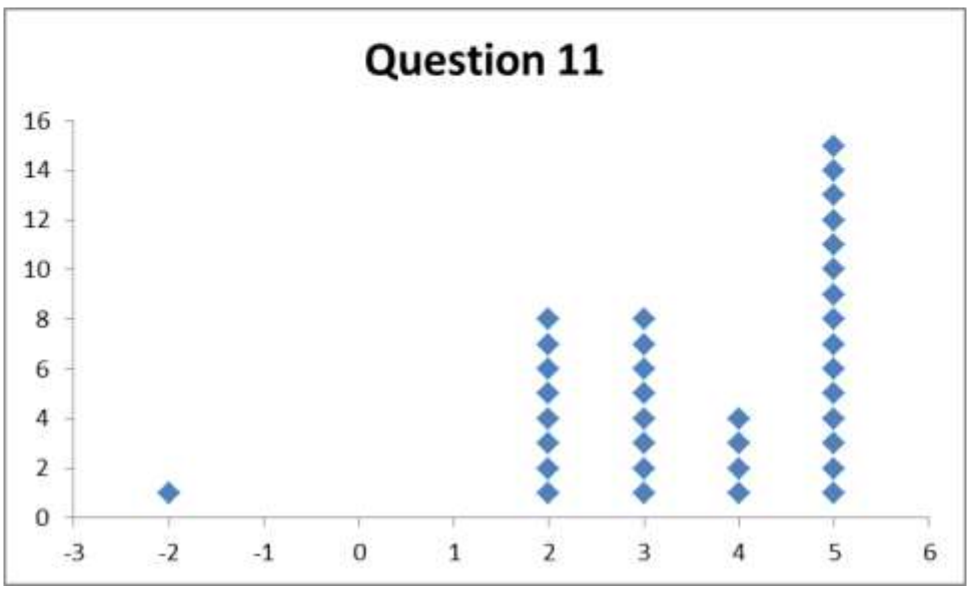
For the problems on this page, consider the relation R on $\mathbb{Z}$ defined by xRy if and only if $x^3 = y^3$. Choose and complete THREE of these problems, and THREE only. If more answers are given, only the first THREE will be graded. (-5/+5 points each)

For each of these questions, note that when you are proving a relation satisfies these properties, you are proving that a certain statement is true *for all* points in the set. This must be done with arbitrary values, say x, y, and z. I repeat, because we've said it about a dozen times in class any many people are still making this mistake: ***you cannot prove something is always true by computing one or two or even a million examples.***

To prove that it is false, however, you need only find one counterexample.

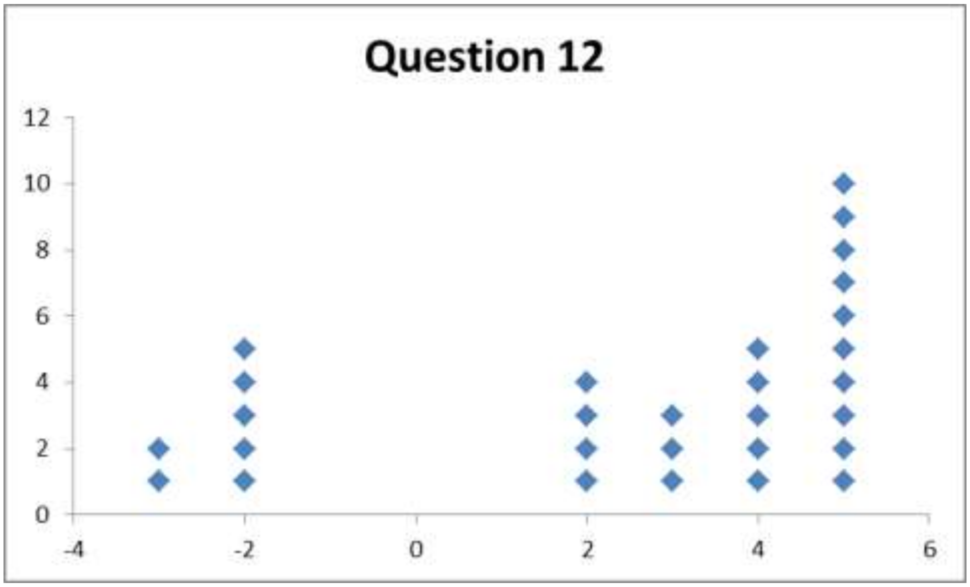
11) Is R reflexive? Justify your answer.

R is reflexive because for any integer x, $x^3 = x^3$.



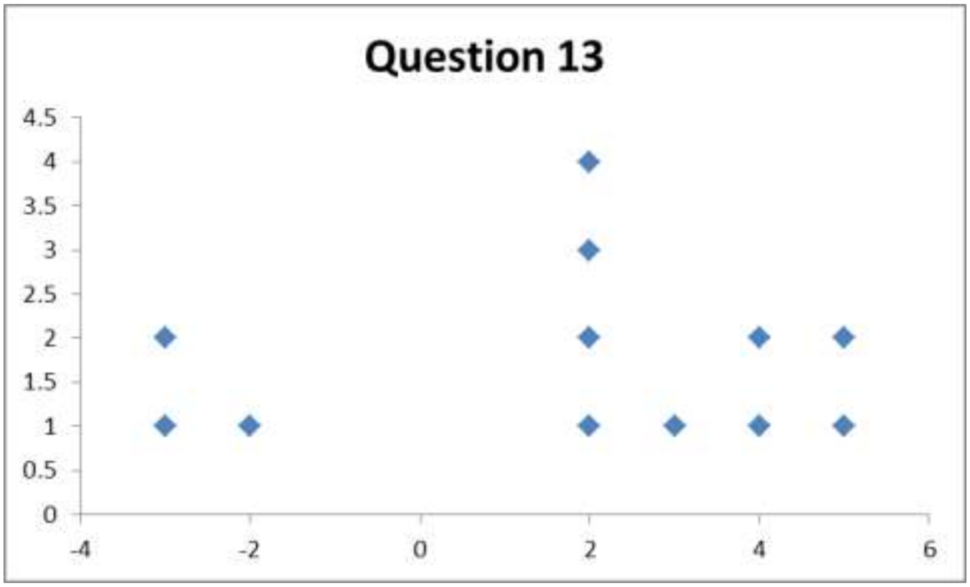
12) Is R symmetric? Justify your answer.

R is symmetric because for any integers x and y, we know that if $x^3 = y^3$ then $y^3 = x^3$.



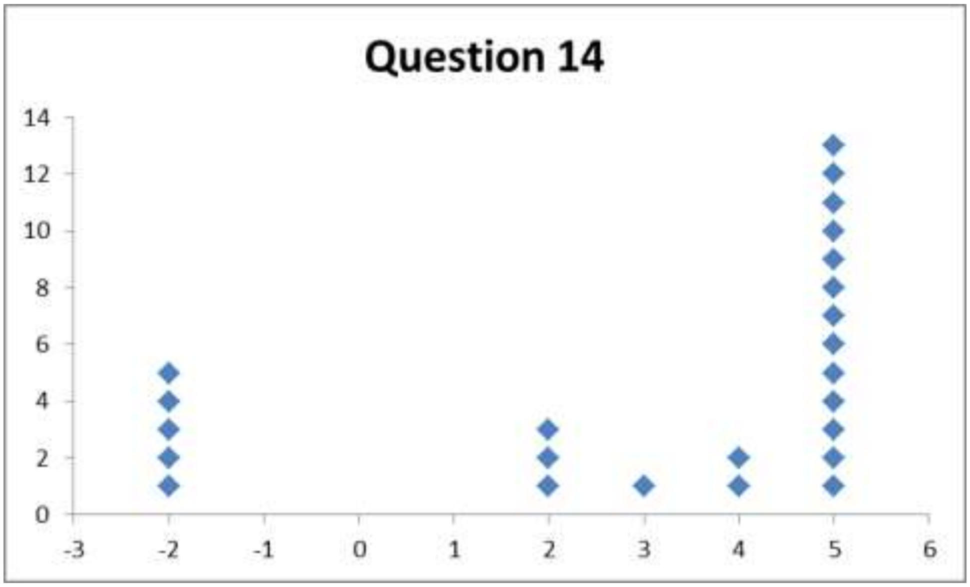
13) Is R antisymmetric? Justify your answer.

R is antisymmetric because if we have two integers x and y such that both $x^3 = y^3$ and $y^3 = x^3$, then we can take the cube root to get $\sqrt[3]{x^3} = \sqrt[3]{y^3}$. This simplifies to $x = y$.



14) Is R transitive? Justify your answer.

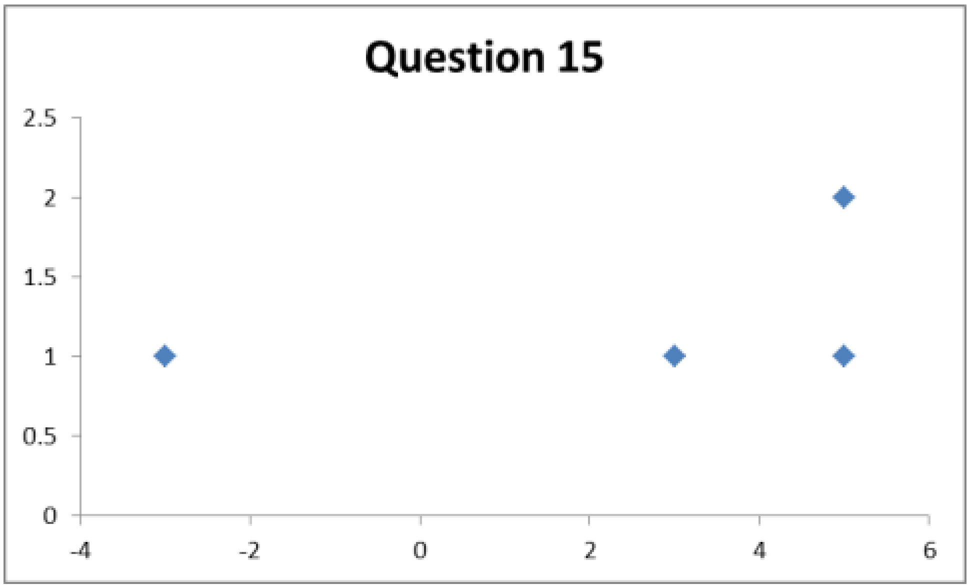
R is transitive because if we know that both $x^3 = y^3$ and $y^3 = z^3$, then we can substitute one form of y^3 into the other to get $x^3 = z^3$.



15) Is R total? Justify your answer.

R is not total because there are two numbers that are not comparable in either direction. In particular 0 and 1. Neither $0^3 = 1^3$ nor $1^3 = 0^3$.

You *must* check both. That's the point of total, that neither xRy nor yRx.



16) Is R irreflexive? Justify your answer.

R is not irreflexive because there is a number that is not not related to itself. In particular 5. $5^3 = 5^3$ and so $5R5$. Hence 5 is not, not related to itself.

Choose and complete ONE of these problems, and ONE only. If more answers are given, only the first ONE will be graded. (-10/+10 points)

17) Define a relation S on $\mathbb{Z}$ by xSy if and only if x divides y. Show that S is a partial order relation.

This question was a little trickier than I intended because of 0. While we cannot divide 0 by 0, it is true that 0 divides 0, because $0 \cdot k = 0$. It's a quirky difference between the division operator (that returns a number) and the divides operator (that returns a Boolean).

It also gets messy dealing with negative numbers. So let's just say that we're only going to deal with positive numbers.

(i.e. I didn't take off points for struggling or overlooking zero or negative numbers)

First let us flesh out this relation a bit. It is given that xSy if and only if $x|y$. This means that there is an integer k such that $xk = y$.

Reflexive: S is reflexive because we can always choose $k = 1$ to get $x \cdot 1 = x$, so $x|x$.

Antisymmetric: S is antisymmetric because if we have both $x|y$ and $y|x$, then that gives the following equations:

$$xk_1 = y$$
$$yk_2 = x$$

Combining these we see that $xk_1k_2 = x$. Cancel x to obtain $k_1k_2 = 1$. But both k_1 and k_2 are positive integers, and the only product of integers that gives 1 is $1 \cdot 1 = 1$. Hence $x = y$.

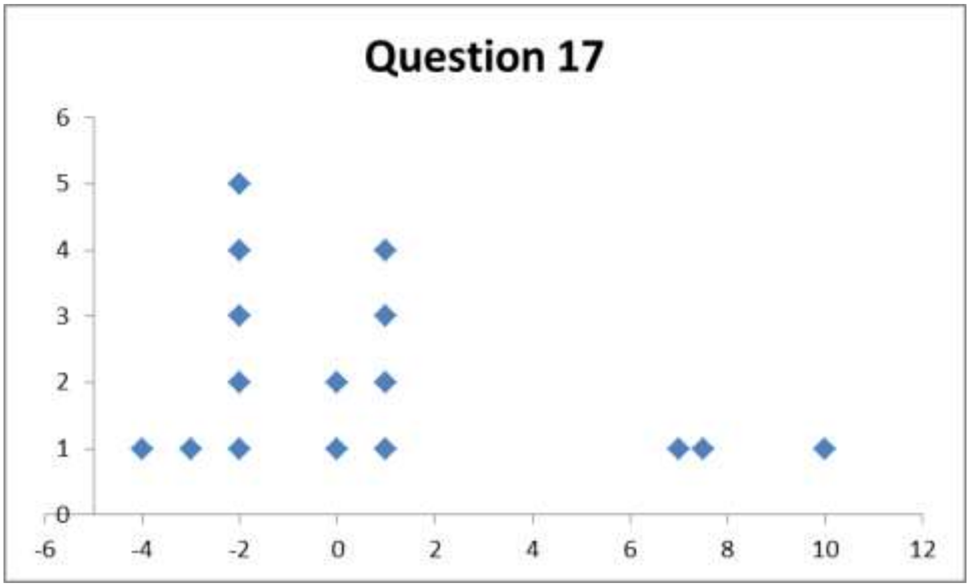
Transitive: S is transitive because if we have both $x|y$ and $y|z$, then that gives the following equations:

$$xk_1 = y$$
$$yk_2 = z$$

Plugging one into the other we get:

$$xk_1k_2 = z$$

Hence $x|z$.


Question 17
6
5
4
3
2
1
0
-6
-4
-2
0
2
4
6
8
10
12

18) Show that the function f, below, is both one to one and onto.

$$f: \mathbb{R} \to \mathbb{R}$$
$$x \to 5x + 7$$

For one-to-one, recall that this means that every one y-value comes from only one x-value. Algebraically this is the following logical statement:

$$\forall_{x_1, x_2 \in \mathbb{R}} (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$$

$$f(x_1) = f(x_2)$$
$$\therefore 5x_1 + 7 = 5x_2 + 7$$
$$5x_1 = 5x_2$$
$$x_1 = x_2$$

For onto, recall that this means that every y-value is attained by some x-value. Algebraically this is the following logical statement:

$$\forall_{y \in \mathbb{R}} \exists_{x \in \mathbb{R}} (f(x) = y)$$

** Do scratch work to figure out that you want to choose $x = \frac{y-7}{5}$ **

Choose $x = \frac{y-7}{5}$ and simplify $f(x)$:

$$f(x) = f\left(\frac{y-7}{5}\right) = 5\left(\frac{y-7}{5}\right) + 7 = (y - 7) + 7 = y$$