Codename __Solutions________________________________________________ Transitions, Test 1

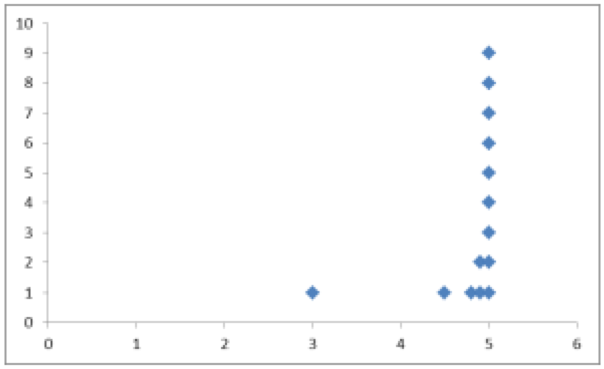
(Do not put your name on the test; write your name and codename on the code sheet)

1) Graham's number, $g_{64}$, is a very very large real number. Consider the statement $\exists_{x\in\mathbb{R}}(x > g_{64})$.

a) Write this statement as an English sentence.

There is a real number larger than Graham's number.

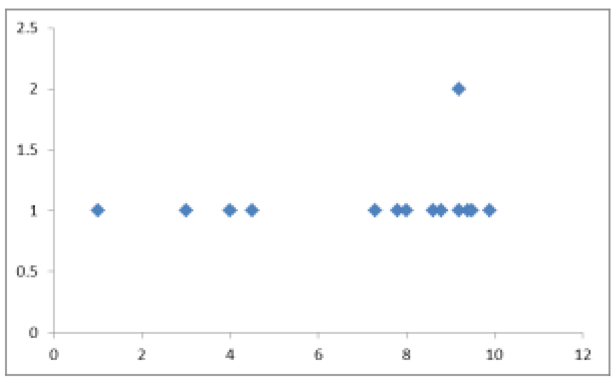
There exists a real number that is larger than Graham's number.




b) Prove this statement.

Let $g_{64}$ denote Graham's number. Choose $x = g_{64} + 1$. Then $x = g_{64} + 1 > g_{64}$.

Therefore there is a real number larger than Graham's number.

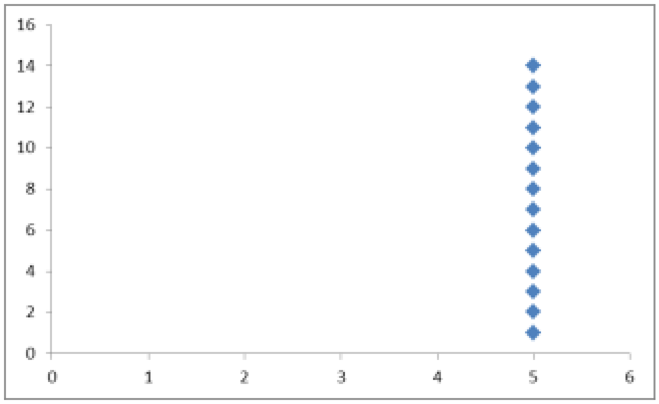


2) Consider the statement $\forall_{x<0}(x^2 > 0)$.

a) Write this statement as an English sentence.

The square of all negative numbers is positive.

For all negative numbers, their square is positive.




b) Prove this statement. (Hint: if $y < 0$, then $y = -|y|$)

Let $x < 0$ be given. Because $x$ is negative, $x = -|x|$. Hence $x^2 = (-|x|)^2 = |x|^2 > 0$. Therefore the square of all negative numbers is positive.

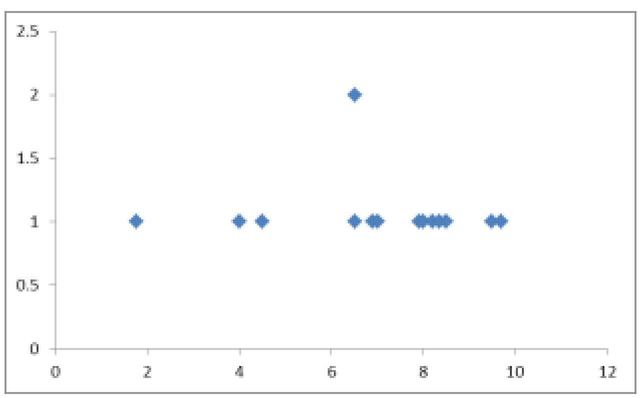
Or written out to emphasize the construction of the implication:

Let $x < 0$ be given.

$\therefore x = -|x|$

$\therefore x^2 = (-|x|)^2 = |x|^2 > 0$

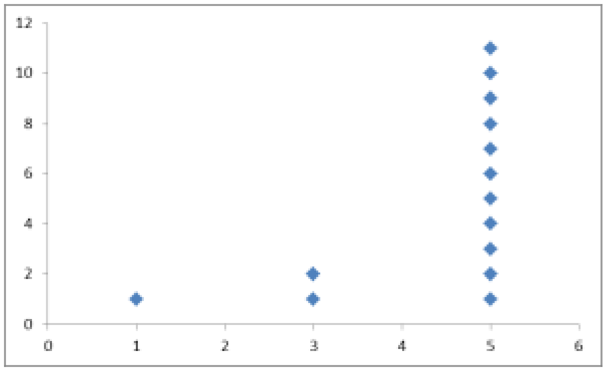
$\therefore \forall_{y<0}(y^2 > 0)$



3) Consider the statement “If a set is a subset of the empty set, then that set is empty.”

a) Write this statement in mathematical notation.

$$A \subseteq \emptyset \Rightarrow A = \emptyset$$




b) Prove this statement.

Let $A$ be a set such that $A \subseteq \emptyset$. Assume $x \in A$. Then also $x \in \emptyset$. However, $x \notin \emptyset$. Hence $x \notin A$. Therefore $A = \emptyset$, and so we have proven that $A \subseteq \emptyset \Rightarrow A = \emptyset$.

Or written out to emphasize the construction of the implication:

Let $A$ bet a set such that $A \subseteq \emptyset$

  Assume $x \in A$

    $\therefore x \in \emptyset$

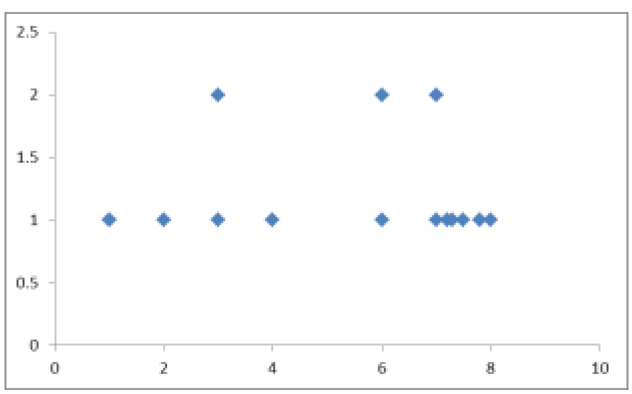
    $x \notin \emptyset$

    $\therefore x \in \emptyset \wedge x \notin \emptyset$

  $\therefore x \notin A$

$\therefore A \subseteq \emptyset \Rightarrow x \notin A$

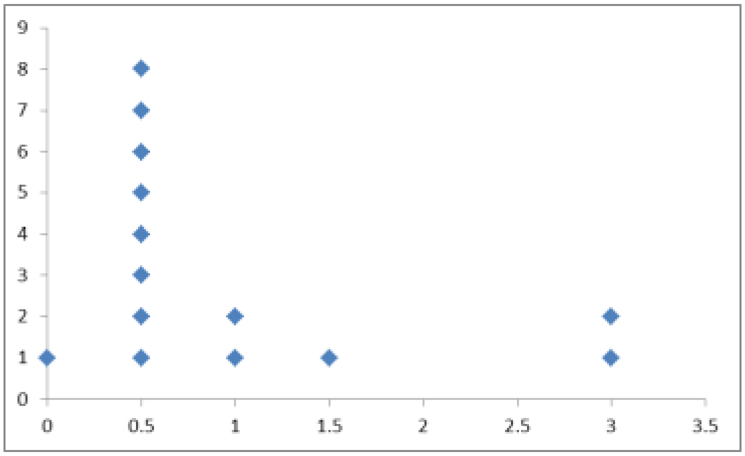
$\therefore A \subseteq \emptyset \Rightarrow A = \emptyset$



4) Define a <u>singleton</u> to be a set with a single element, such as $\{1\}$. Consider the statement "The cross product of a set with a singleton is the same size as the original set".

a) Write this statement in mathematical notation.

Let $A$ be a set. $|A \times \{b\}| = |A|$



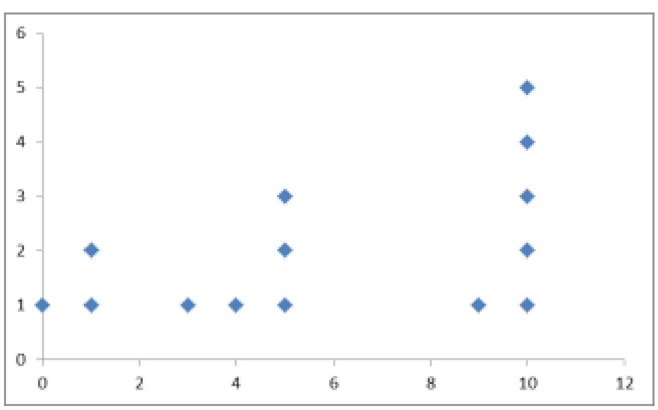

b) Give a sketch of the proof of the statement.

We'll see more on how to formally prove this later (shown below for later reference). For this test a sketch of the idea, $a_i \leftrightarrow (a_i, b)$, was good enough.

" $\leq$ " Let $x \in A \times \{b\}$ be an arbitrary element of the left hand side. Then there is an $a \in A$ such that $x = (a, b)$. But $a \in A$, so we also have an element of the right hand side. Furthermore if $a_1 = a_2$ then $(a_1, b) = (a_2, b)$. Thus $|A \times \{b\}| \leq |A|$.

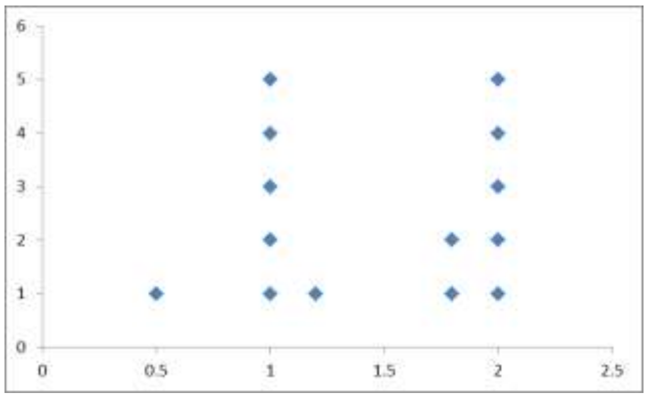
" $\geq$ " Let $a \in A$ be an arbitrary element of the right hand side. Then $(a, b)$ is an element of the left hand side. Furthermore if $(a_1, b) = (a_2, b)$ then $a_1 = a_2$. Thus $|A \times \{b\}| \geq |A|$

Therefore $|A \times \{b\}| = |A|$



c) Which of the following are singletons? Circle them.

{1,2}, {{1,2}}, {1}, {2}, {{1}}, {{2}}, {Ø}, {{Ø}}, {{}}



5) Show the following theorem for all sets $A$:

$$A = \bigcup_{B \in \mathcal{P}(A)} B$$

"⊆" Assume $x \in A$. Then $\{x\} \subseteq A$, so $\{x\} \in \mathcal{P}(A)$. Hence $\{x\} \subseteq \bigcup_{B\in\mathcal{P}(A)} B$ and so $x \in \bigcup_{B\in\mathcal{P}(A)} B$. Thus $A \subseteq \bigcup_{B\in\mathcal{P}(A)} B$

"⊇" Assume $x \in \bigcup_{B\in\mathcal{P}(A)} B$. Then there is some $B \in \mathcal{P}(A)$ such that $x \in B$. Let's call this particular set $B_j$. That is, $x \in B_j$. Now $B_j \subseteq A$, so $x \in A$ also. Thus $A \supseteq \bigcup_{B\in\mathcal{P}(A)} B$

Therefore :

$$A = \bigcup_{B \in \mathcal{P}(A)} B$$

Or written out to emphasize the construction of the implication:

Assume $x \in A$.

$\therefore \{x\} \subseteq A$

$\therefore \{x\} \in \mathcal{P}(A)$.

$\therefore \{x\} \subseteq \bigcup_{B\in\mathcal{P}(A)} B$

$\therefore x \in \bigcup_{B\in\mathcal{P}(A)} B$.

$\therefore A \subseteq \bigcup_{B\in\mathcal{P}(A)} B$

Assume $x \in \bigcup_{B\in\mathcal{P}(A)} B$.

$\therefore \exists_{B\in\mathcal{P}(A)}(x \in B)$. Label such a set $B_j$.

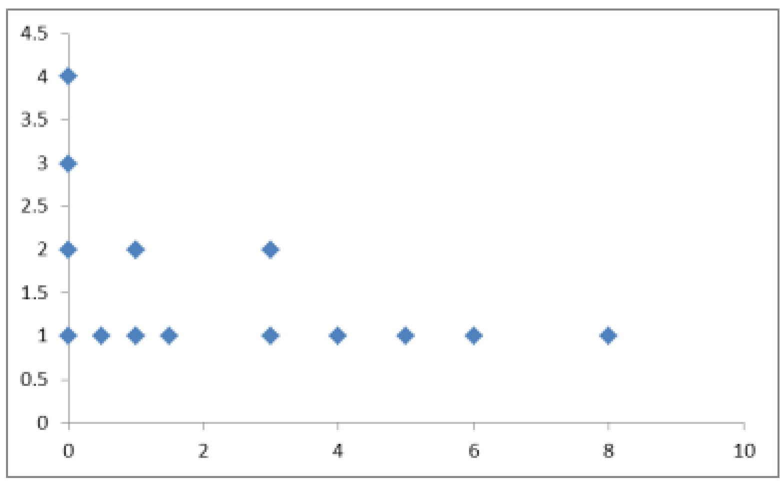
$\therefore x \in B_j$.

$\therefore x \in B_j$.

$B_j \subseteq A$.

$\therefore x \in A$

$\therefore A \supseteq \bigcup_{B\in\mathcal{P}(A)} B$



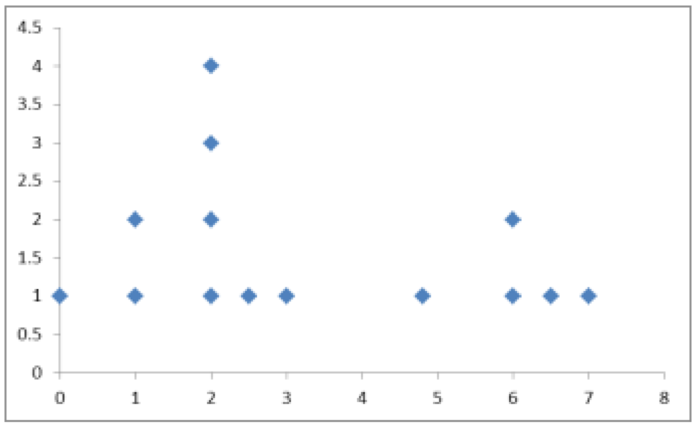

Therefore

$$A = \bigcup_{B \in \mathcal{P}(A)} B$$

6) Show the following theorem for all sets $A, B$, and $C$.

$$(A - B) - C = A - (B \cup C)$$

$$\begin{aligned}(A - B) - C &= (A - B) \cap C^c \\ &= (A \cap B^c) \cap C^c \\ &= A \cap (B^c \cap C^c) \\ &= A \cap (B \cup C)^c \\ &= A - (B \cup C)\end{aligned}$$



7) Show the following theorem for all statements $P$ and $Q$:

$$(P \Leftrightarrow Q) \Rightarrow (\sim P \vee Q)$$

| $P$ | $Q$ | $P \Leftrightarrow Q$ | $\sim P$ | $\sim P \vee Q$ | $(P \Leftrightarrow Q) \Rightarrow (\sim P \vee Q)$ |
|---|---|---|---|---|---|
| T | T | T | F | T | T |
| T | F | F | F | F | T |
| F | T | F | T | T | T |
| F | F | T | T | T | T |

Because the final column for the implication is all true's, this is a tautology and so this statement is a theorem.

OR

Let $P$ and $Q$ be statements. Assume $P \Leftrightarrow Q$. Then in particular $P \Rightarrow Q$, which is equivalent to $\sim P \vee Q$. Therefore $(P \Leftrightarrow Q) \Rightarrow (\sim P \vee Q)$.

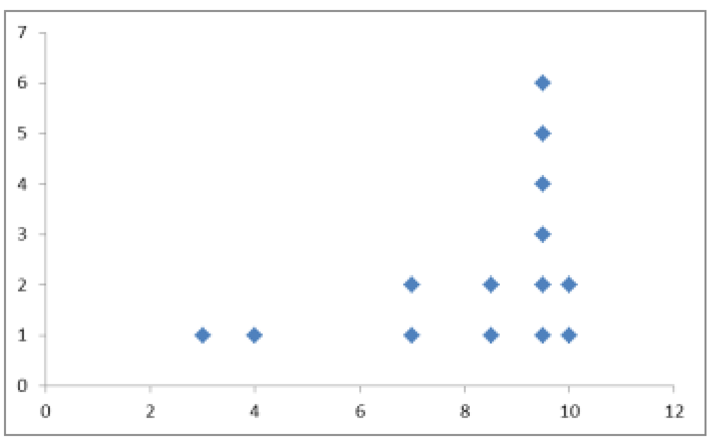
Or written out to emphasize the construction of the implication:

Assume $P \Leftrightarrow Q$

$\quad \therefore P \Rightarrow Q$

$\quad \therefore \sim P \vee Q$

$\therefore (P \Leftrightarrow Q) \Rightarrow (\sim P \vee Q)$



8) Show the following theorem:

$$\forall_{x>0}\exists_{y>0}\forall_{z>0}[(z<y)\Rightarrow(3z<x)]$$

Let $x>0$ and choose $y=\frac{x}{3}$. Further let $z>0$ and assume $z<y$. Then substituting in for $y$ we have $z<\frac{x}{3}$ which is the same as $3z<x$. Putting this all together this proves the theorem: For all $x>0$ we may choose $y=\frac{x}{3}$ so that for all $z>0, 3z<x$ whenever $z<y$.

Or written out to emphasize the construction of the implication:

Let $x>0$.

  Choose $y=\frac{x}{3}$.

  Let $z>0$.

    Assume $z<y$

      $\therefore z<\frac{x}{3}$

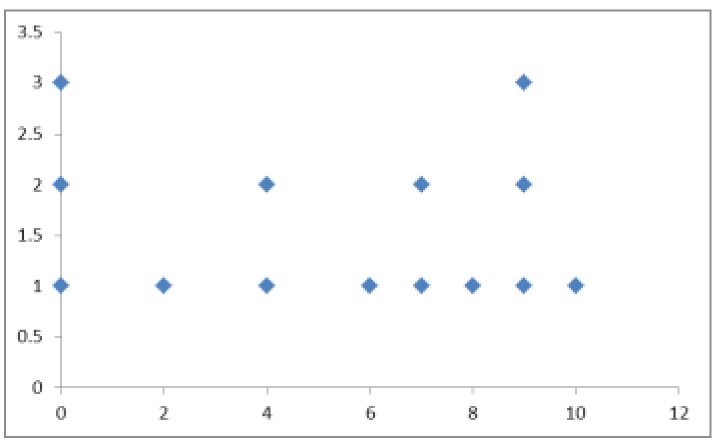
      $\therefore 3z<x$

    $\therefore z<y\Rightarrow 3z<x$

  $\therefore \forall_{z>0}(z<y\Rightarrow 3z<x)$

  $\therefore \exists_{y>0}\forall_{z>0}(\forall_{z>0}(z<y\Rightarrow 3z<x)$

$\therefore \forall_{x>0}\exists_{y>0}\forall_{z>0}(\forall_{z>0}(z<y\Rightarrow 3z<x)$



Grading Notes: Question 8 used a different scale than the rest of the proofs: 2 points were awarded for each of the following and no points were deducted for other mistakes:

- Fixing $x$ arbitrarily
- Choosing $y$ (half credit if an incorrect choice)
- Fixing $z$ arbitrarily
- Assuming $z < y$
- Correctly forming the implication $z < y \Rightarrow 3z < x$

Half credit was given for any of the first four above if they were in the wrong order.

You can interpret scores on the other proofs as follows:

10 – Perfectly correct.
8 – All the right ideas with minor mistakes.
6 – On the right track with one or two significant issues.
3 – Had an idea that could work, but it's not clear if you know how to make it work.
1 – Showed some understanding of something, but there is no evidence you can make it into a proof.

Green marks correspond to phrases that do not make mathematical sense.
Blue marks correspond to phrases that are mathematically incorrect.
Orange marks correspond to phrases that are mathematically correct, but unjustified.
Red marks are anything else.