Codename __Solutions____ Transitions, Test 2

(Do not put your name on the test; write your name and codename on the code sheet)

1) Show that for all $n \in \mathbb{Z}_{\geq 1}$:

$$\sum_{i=1}^{n} i \cdot 2^i = 2 + (n-1)2^{n+1}$$

Because of the universal I'm thinking induction might be a good idea. With the summation, I see a clear direction to go because I know I'll be able to use the induction hypothesis.

Base Case: For $n = 1$ the left hand side is:

$$\sum_{i=1}^{i} i \cdot 2^i = 1 \cdot 2^1 = 2$$

The right hand side is:

$$2 + (1-1)2^{1+1} = 2 + 0 = 2$$

These are equal, and so the base case is satisfied.

Induction Hypothesis: Assume for some $k \in \mathbb{N}$ that

$$\sum_{i=1}^{k} i \cdot 2^i = 2 + (k-1)2^{k+1}$$

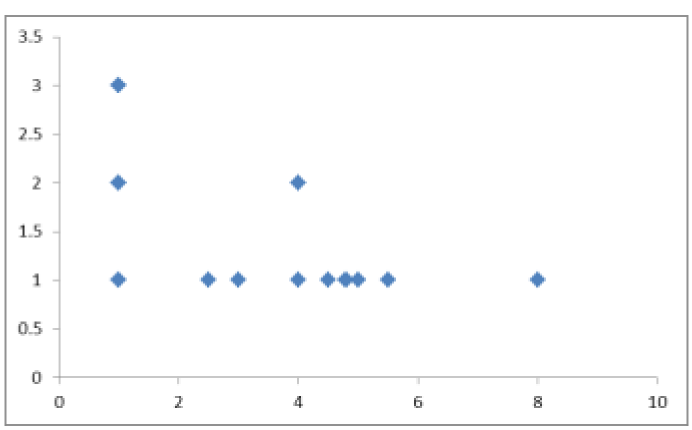
Induction Step: Indeed the "$k+1$" case is satisfied:

$$\begin{aligned}\sum_{i=1}^{k+1} i \cdot 2^i &= \left(\sum_{i=1}^{k} i \cdot 2^i\right) + \left((k+1)2^{k+1}\right) \\ &= 2 + (k-1)2^{k+1} + (k+1)2^{k+1}n \\ &= 2 + 2k2^{k+1} \\ &= 2 + k2^{k+2} \\ &= 2 + \left((k+1) - 1\right)2^{(k+1)+1}\end{aligned}$$

Thus $\sum_{i=1}^{k+1} i \cdot 2^i = 2 + \left((k+1)-1\right)2^{(k+1)+1}$ which is the "$k+1$" case.

Therefore by induction the statement holds true for all $n \in \mathbb{N}$. That is:

$$\forall_{z \in \mathbb{N}} \left(\sum_{i=1}^{n} i \cdot 2^i = 2 + (n-1)2^{n+1}\right)$$



2) Let A be the set of all people. Choose ONE of the following relations and show that it is an equivalence relation:
R_1 is the relation on A such that xRy if and only if x and y have the same shoe size.
R_2 is the relation on A such that xRy if and only if x and y are either both male, or both female.

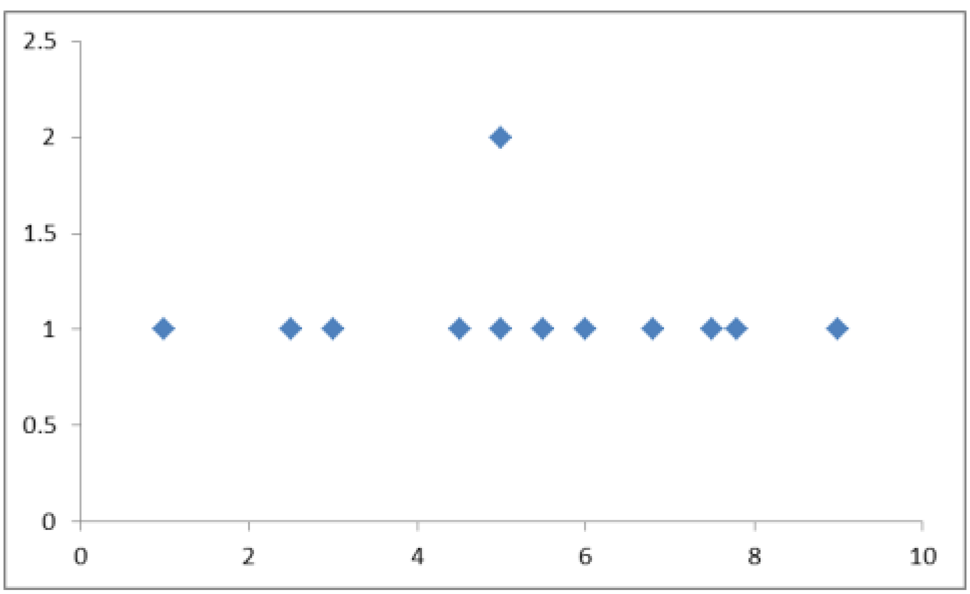
We are given that R is a relation, so we need only show that R is reflexive, symmetric, and transitive.

Shoe size:
Reflexive: xRx because x certainly has the same shoe size as himself.
Symmetric: Assume xRy. That is to say that x and y have the same shoe size. Rewording this we can say that y and x have the same shoe size, so yRx.
Transitive: Assume xRy and yRz. That is to say that x and y have the same shoe size, and also that y and z have the same shoe size. Thus all three have the same shoe size as y, so in particular x and z have the same shoe size: xRz.

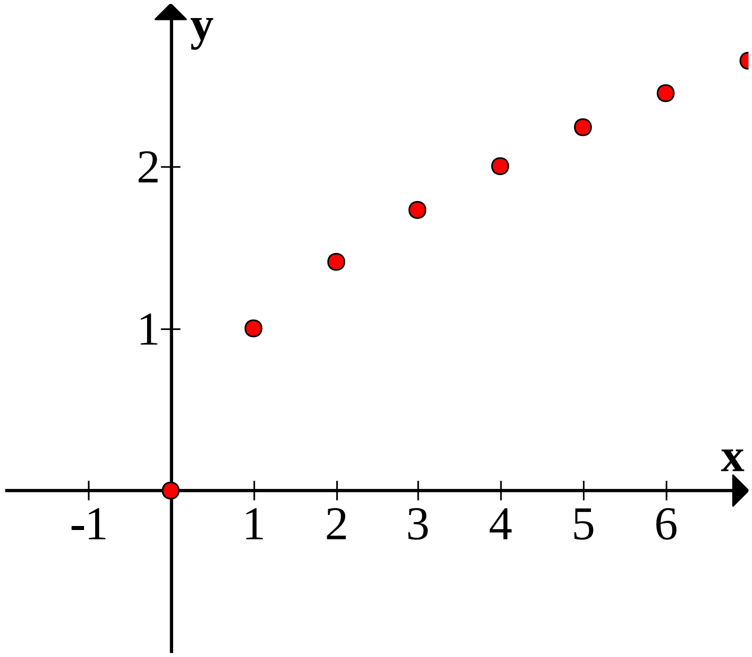
Gender:
Reflexive: xRx because x certainly has the same gender as himself.
Symmetric: Assume xRy. That is to say that x and y have the same gender. Rewording this we can say that y and x have the same gender, so yRx.
Transitive: Assume xRy and yRz. That is to say that x and y have the same gender, and also that y and z have the same gender. Thus all three have the same gender as y, so in particular x and z have the same gender: xRz.

Therefore R is an equivalence relation.

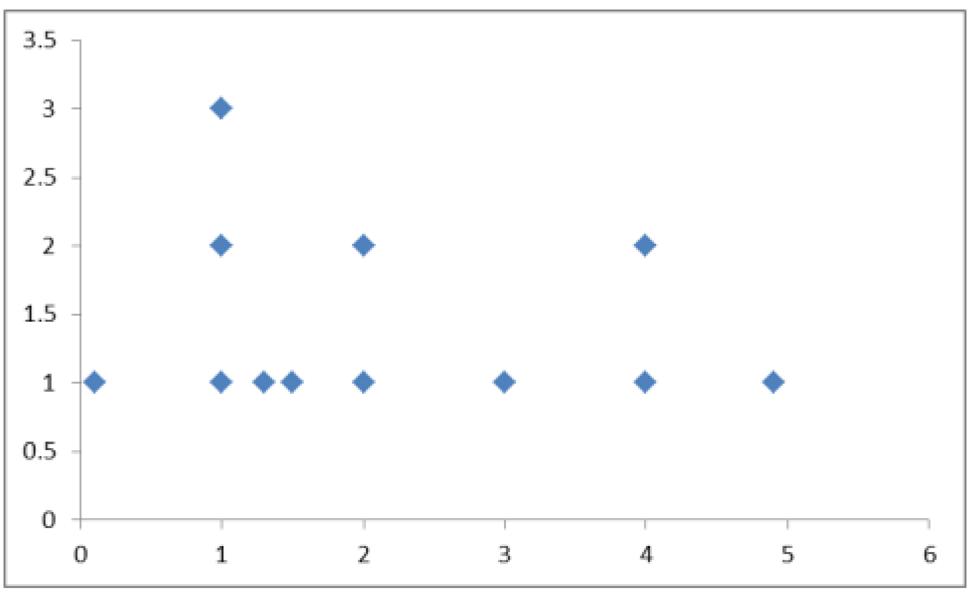
Be careful that here x, y, and z are people. Yes they have a shoe size and gender, but they are people.



3) Let $f: \mathbb{Z} \to \mathbb{R}$ be the relation given by $f(x) = \sqrt{x}$ when possible. Sketch a graph of f then prove or disprove that f is a function.



f is not a function because its domain is not $\mathbb{Z}$. In particular, $f(-1) = \sqrt{-1} = i \notin \mathbb{R}$.



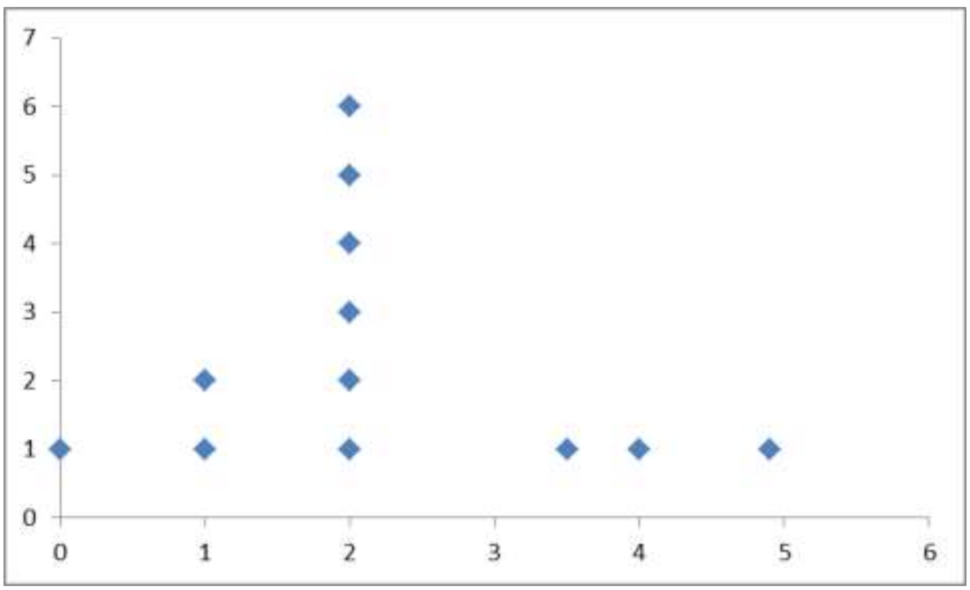
4) Let $f: \mathbb{R} \to \mathbb{C}$ be the relation given by $f(x) = \sqrt{x}$. Prove or disprove that f is a function.

f is a function. By construction f is a relation, it remains to be proven that f's domain is $\mathbb{R}$ and that it is well defined.

Domain:
Let $x \in \mathbb{R}$. Then the square root of x is some complex number. Hence $f(x) \in \mathbb{C}$. (Actually, either $f(x) \in \mathbb{R}$ or $f(x) \in i\mathbb{R}$, but either way $f(x)$ is a complex number).

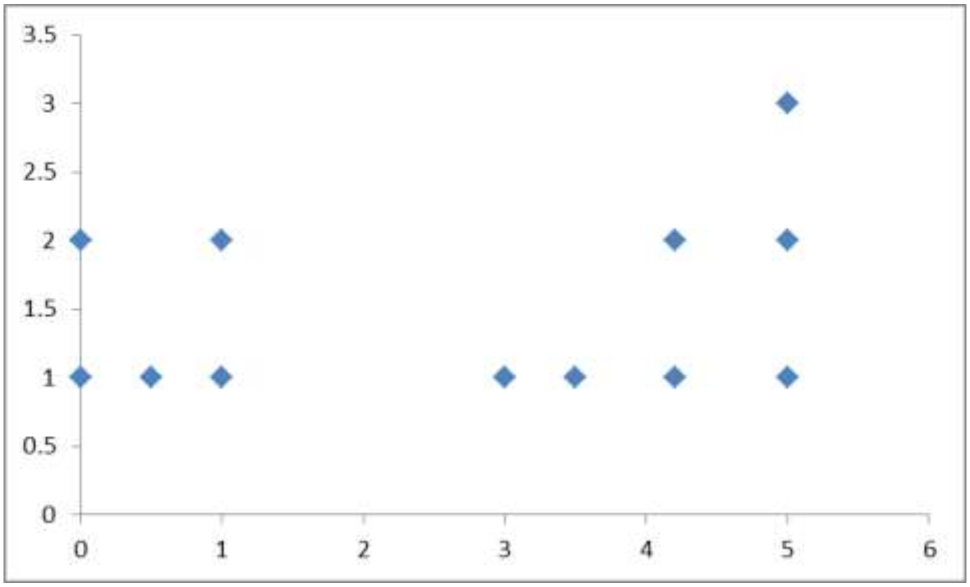
Well Defined:
Given a complex number x, $\sqrt{x}$ represents one specific number (the principal square root), hence $f(x)$ is just one number and so f is well defined.

Thus f is a function.

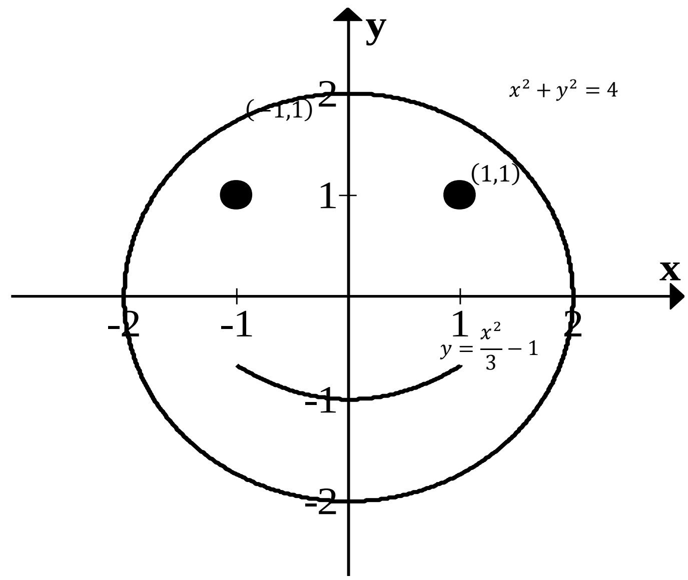


5) Let $f: \mathbb{R} \to \mathbb{C}$ be the relation given by $f(x) = \pm\sqrt{x}$. Prove or disprove that f is a function.

f is not a function because it is not well defined. In particular $f(4) = \pm 2$. Whaaat? $f(4) = 2$ but also $f(-4) = -2$?! This f is not a function.

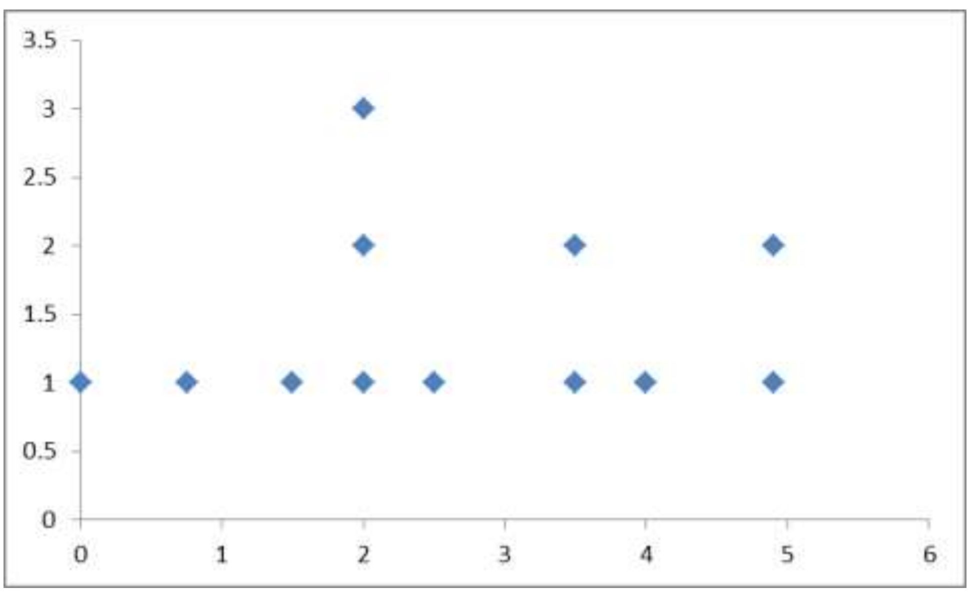


6) Let R be the graph of the entire smiley face below. Is R a relation? If so what is it as a set? If not, why not?

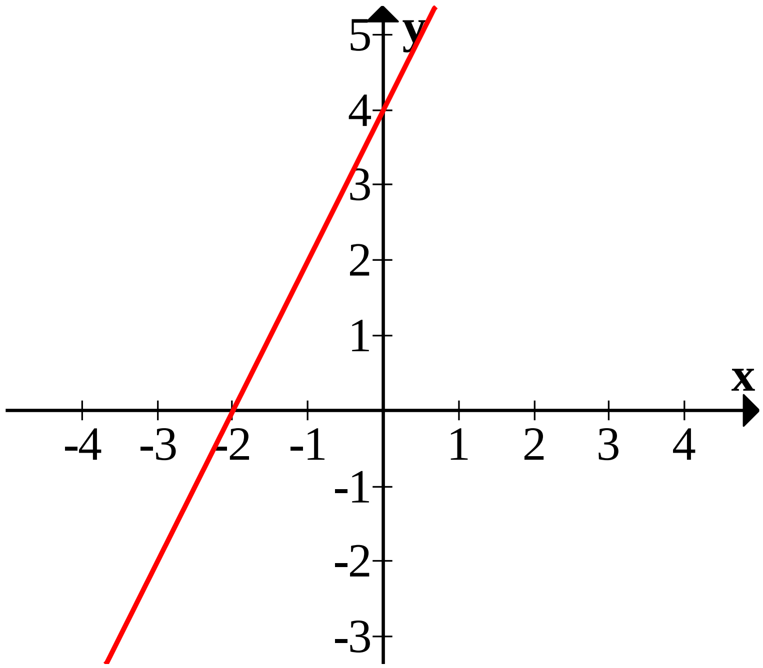


Yes, R is a relation:

$$R = \left\{(x,y) \middle| x^2 + y^2 = 4 \text{ or } y = \frac{x^2}{3} - 1 \text{ or } (x,y) = (-1,1) \text{ or } (x,y) = (1,1)\right\}$$



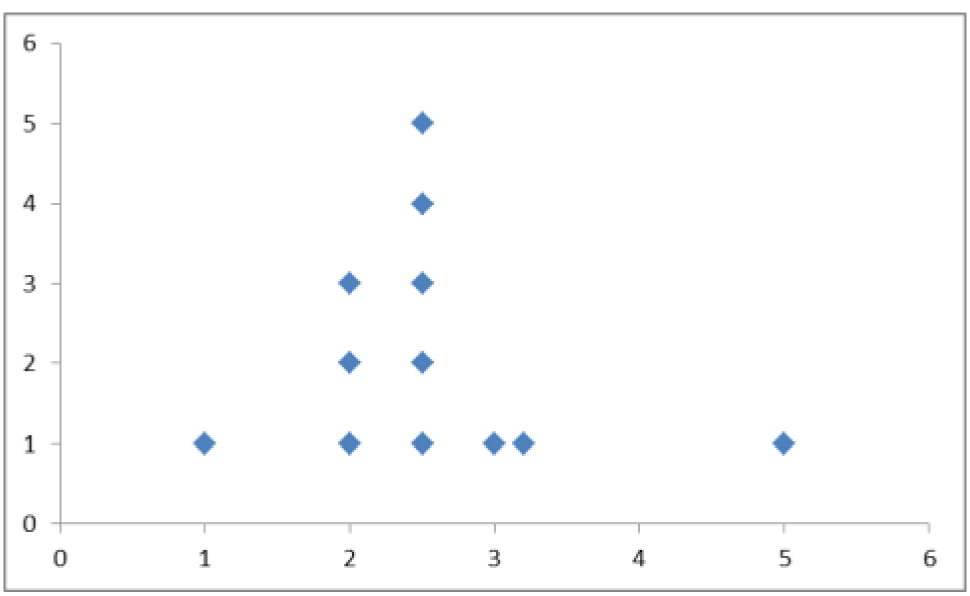
7) Let $f: \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = 2x + 4$. Sketch a graph of f and prove or disprove that f is one-to-one.



Indeed f is one-to-one:

Suppose $f(a) = f(b)$ for some $a, b \in \mathbb{R}$. Then:

$$2a + 4 = 2b + 4$$
$$\therefore 2a + 2b$$
$$\therefore a = b$$

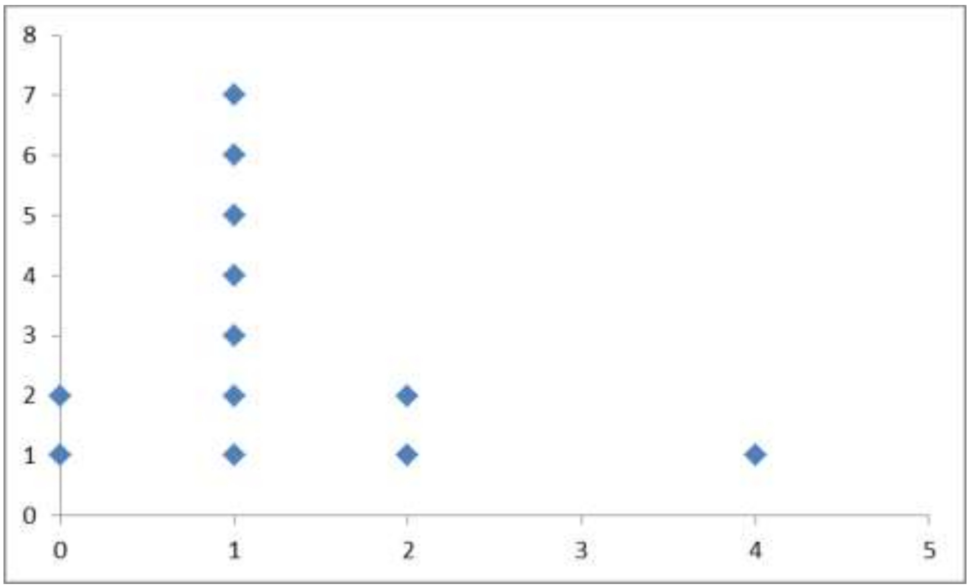


8) Let $f: \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = 2x + 4$. Prove or disprove that f is onto.

Indeed f is onto:

Suppose $y \in \mathbb{R}$. Then choose $x = \frac{y-4}{2}$. Then:

$$f(x) = f\left(\frac{y-4}{2}\right) = \left(\frac{y-4}{2}\right)2 + 4 = y$$



9) Let A be a set, and define a binary relation $\rtimes$ on A. (For example, $\mathbb{R}$ and addition satisfy this). Now suppose that $\rtimes$ is actually associative: $a \rtimes b \rtimes c$ is unambiguous in that $(a \rtimes b) \rtimes c = a \rtimes (b \rtimes c)$ for all $a, b, c \in A$. Sketch a proof of the fact that for any $n \in \mathbb{Z}_{\geq 3}$:

$$a_1 \rtimes a_2 \rtimes \cdots \rtimes a_n \text{ is unambiguous.}$$

The universal makes me think that induction might work on this.

Base case: The base case is given to us as $\rtimes$ is associative.

Induction hypothesis: Assume for some $k \in \mathbb{Z}_{\geq 3}$ that $a_1 \rtimes a_2 \rtimes \cdots \rtimes a_k$ is unambiguous.

Induction step: Now consider $a_1 \rtimes a_2 \rtimes \cdots \rtimes a_{k+1}$. There are k different groupings to consider:

$$(a_1 \rtimes \cdots \rtimes a_l) \rtimes (a_{l+1} \rtimes \cdots \rtimes a_{k+1})$$

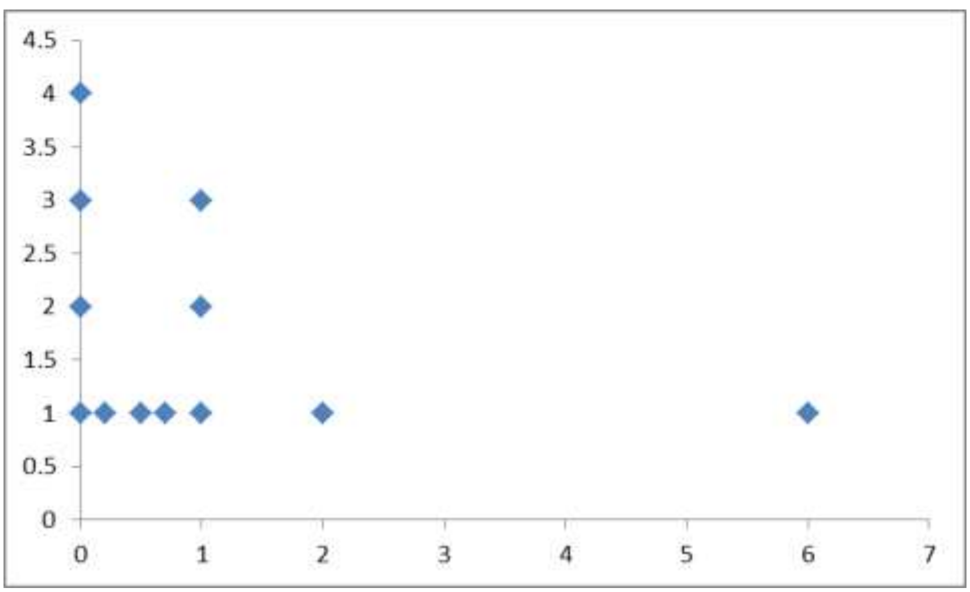
That is, above l could be any of $1, 2, \dots, k$.

Now because of the inductive hypothesis each parenthesized portion is unambiguous. Then by associativity we may regroup it to include one more term in the left portion:

$$(a_1 \rtimes \cdots \rtimes a_l) \rtimes \left(a_{l+1} \rtimes (a_{l+2} \rtimes \cdots \rtimes a_{k+1})\right) = \left((a_1 \rtimes \cdots \rtimes a_l) \rtimes a_{l+1}\right) \rtimes (a_{l+2} \rtimes \cdots \rtimes a_{k+1})$$

We see then taking $l = 1$ we may regroup to obtain the $l = 2$ parenthesization, and then regroup again to obtain the $l = 3$ parentheisization. By doing this $k - 1$ times we have all the different parenthisizations, and so $a_1 \rtimes a_2 \rtimes \cdots \rtimes a_{k+1}$ is unambiguous!

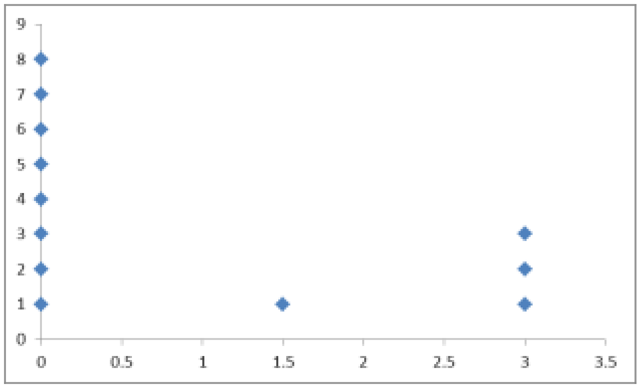
Therefore by induction any $n \in \mathbb{Z}_{\geq 3}$, $a_1 \rtimes a_2 \rtimes \cdots \rtimes a_n$ is unambiguous.



Consider the following function diagram:

$$\begin{array}{ccccc} \mathbb{R} & \overset{f_1}{\rightarrow} & \mathbb{R} & \overset{f_2}{\rightarrow} & \mathbb{R} \\ f_3 \downarrow & \circlearrowleft & \downarrow f_4 & & \downarrow f_5 \\ \mathbb{Q} & \underset{f_6}{\rightarrow} & \mathbb{Z} & \underset{f_7}{\rightarrow} & \mathbb{R} \end{array}$$

10) Find functions $f_1, f_2, \ldots, f_7$ such that the diagram is commutative in the left square, but not in the right square.



There are many answers, I'll try to choose very simple ones:

$f_1(x) = x$

$f_2(x) = x$

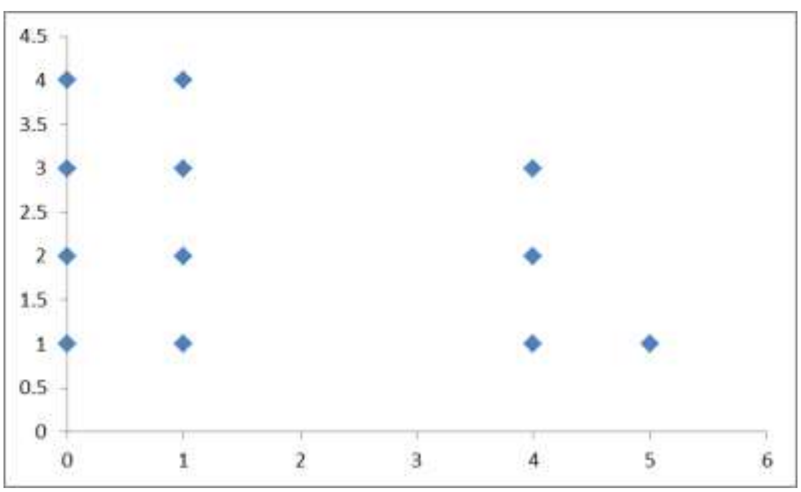
$f_3(x) = 1$

$f_4(x) = 1$

$f_5(x) = x$

$f_6(x) = x$

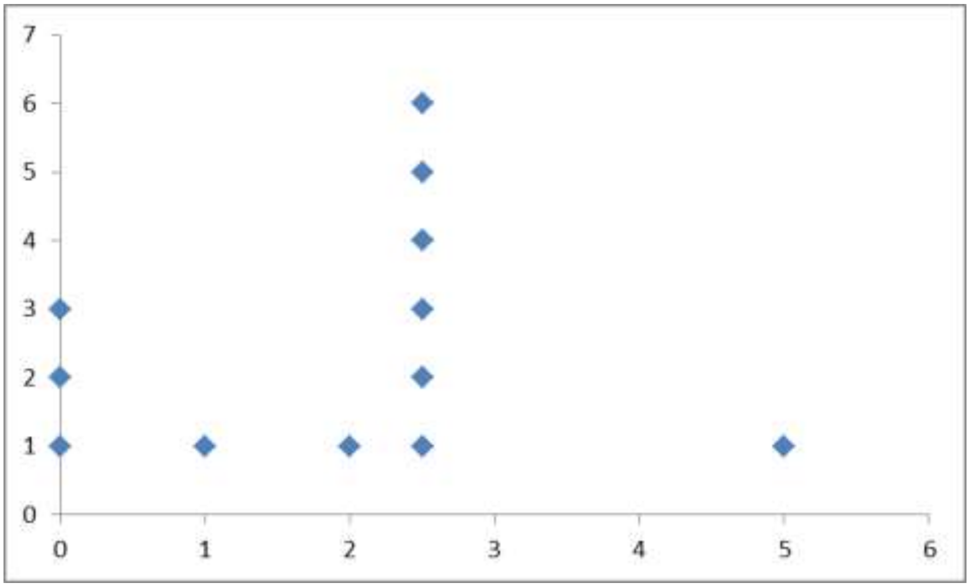
$f_7(x) = x$



11) Is it true that $f_4 \circ f_1 = f_6 \circ f_3$? Why?

Yes, because:

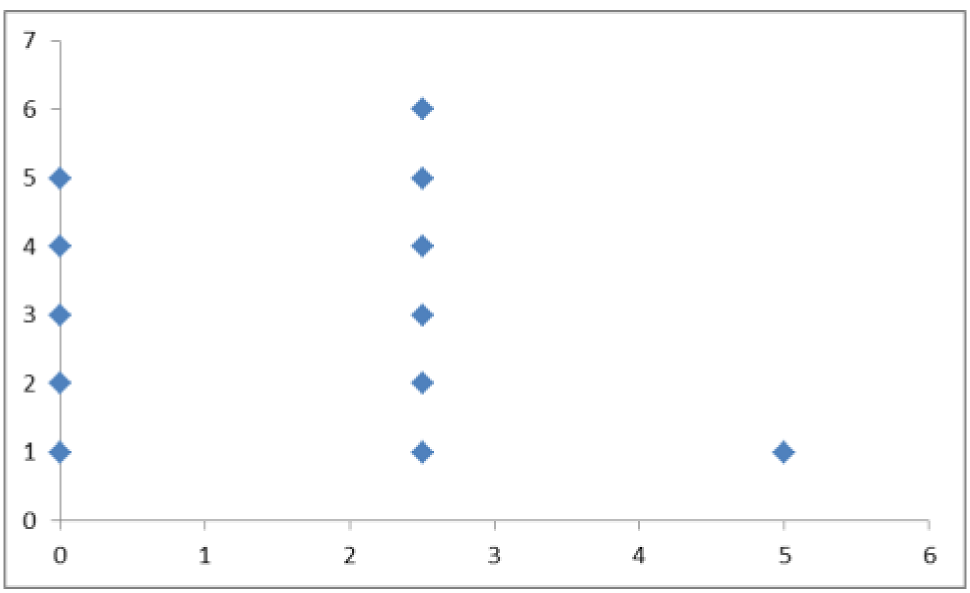
$$f_4(f_1(x)) = 1$$
$$f_6(f_3(x)) = 1$$



12) Is it true that $f_5 \circ f_2 = f_7 \circ f_4$? Why?

No, because:

$$f_5(f_2(x)) = x$$
$$f_7(f_4(x)) = 1$$



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Let A be the set of all monomials involving the variables x and or y. (A monomial is a term consisting of variables to nonnegative integer powers, all multiplied by each other).
For example, the following are all such monomials:

$$x \qquad x^6 \qquad xy^2 \qquad x^{240}y$$

As a nonexample, the following are not elements of A:

$$2x \qquad xy^2z^7t \quad x^{-1}y \qquad x^{2.5}$$

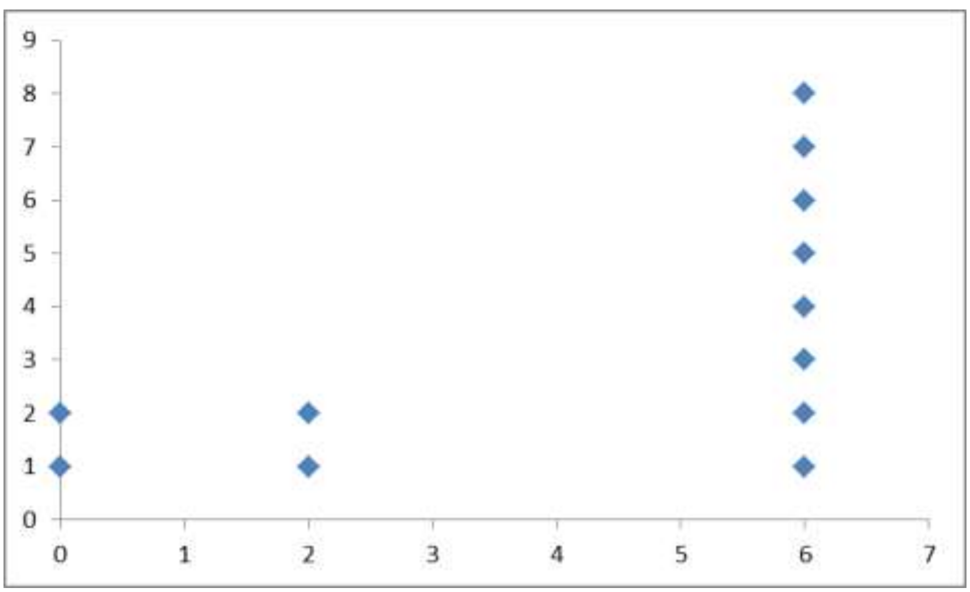
Define the total degree, d, of a monomial as the sum of the exponents.
For example $d(xy^7) = 8$ while $d(x^4y^2) = 6$.

Finally, for monomials a_1 and a_2, define the relation $\prec$ on A via $a_1 \prec a_2$ if and only if one of the following is satisfied:
$d(a_1) < d(a_2)$
OR
$d(a_1) = d(a_2)$ and the degree of x in a_1 is less than the degree of x in a_2.

As one last step, define $\preccurlyeq$ as " $\prec$ or $=$ ".

13) Fill in each of the following boxes with either $\preccurlyeq$ or $\succcurlyeq$:

$$x^2y^6 \preccurlyeq x^3y^6$$
$$x^2y^6 \preccurlyeq x^3y^5$$
$$x^2y^6 \succcurlyeq x^1y^7$$



Prove or disprove each of the following: (Use the back page and clearly label each problem)

14) $\preccurlyeq$ is reflexive.

Proof: Let $a \in A$, wlog $a = x^b y^c$. Indeed comparing $x^b y^c$ to itself we see that the total degree is the same, and the degree of x is the same. Hence $x^b y^c \preccurlyeq x^b y^c$

15) $\preccurlyeq$ is symmetric.

disproof: Consider x and x^2. $x \preccurlyeq x^2$, but $x^2 \not\preccurlyeq x$

16) $\preccurlyeq$ is antisymmetric.

Proof: Suppose $a \preccurlyeq b$ and $b \preccurlyeq a$. Wlog let $a = x^r y^s$ and $b = x^p y^q$. Then $x^r y^s \preccurlyeq x^p y^q$ and also $x^r y^s \succcurlyeq x^p y^q$. If the total degree were different, only one of these would hold. Hence the total degree is the same. Then if the degree of x were different, only one of these would hold. Hence the degree in x is the same.

Now because the total degree is the same, and the degree in x is the same, then the degree in y is the same. That is to say, $r = p$ and $s = q$. Hence $a = x^r y^s = b$.

17) $\preccurlyeq$ is transitive.

Proof: Suppose $a \preccurlyeq b$ and $b \preccurlyeq c$. If in either case the total degree increases ($d(a) < d(b)$ or $d(b) < d(c)$), then the total degree increases from a to c: $d(a) < d(c)$.

On the other hand if the total degree for a, b, and c are the same, then we look at the degree in x. Either the degree of x increases or stays the same. Either way is sufficient, and so we find that $a \preccurlyeq c$.

18) All elements of A are comparable under $\preccurlyeq$.

Indeed this is the case. If their total degree's differ, the smaller one is truly "smaller". If the total degree's are the same we look at the degree in x in which case we find that the smaller degree gives the "smaller" monomial. If, however, the total degree and degree in x are the same, then the monomials are equal, and so again they would be comparable via $\preccurlyeq$. (But not via $\prec$).

19) $\preccurlyeq$ is an equivalence relation.

This is not the case because it is not symmetric.

20) $\preccurlyeq$ is a total ordering.

This is the case because it is reflexive, antisymmetric, transitive, and all elements are comparable.