Name ______Solutions____ Transitions, Quiz 2

1) Sketch an outline of each of the following proof methods. (10 points each)

a. Proof of an equality

Start with one side of equation and manipulate it (simplify, complicate, whatever you can do) to get the other side.

$$LHS = \cdots = RHS$$
$$\therefore LHS = RHS$$

b. Proof of an implication

Assume the sufficient condition. Then show the necessary condition.

Assume P
$\vdots$
Thus Q.
Therefore $P \Rightarrow Q$

c. Proof of a subset

Start with an arbitrary element of the “small” set and show it’s in the “big” set.

Assume $x \in A$
$\vdots$
Thus $x \in B$
Therefore $A \subseteq B$

d. Proof of an if-and-only-if.

We need to show two implications.

Assume P

$\vdots$

Thus Q.

Thus $P \Rightarrow Q$

Assume Q.

$\vdots$

Thus P.

Thus $Q \Rightarrow P$

Therefore $P \Leftrightarrow Q$

e. Proof of an existential

Show that the open statement in question is true for something.

Choose $n =$ blah

$\vdots$

Thus $P(n)$ is true.

f. Proof of a universal

We show that the statement is true for an arbitrary element. Then that generalizes to apply to every element.

Let x be an arbitrary blah.

$\vdots$

Thus [statement in question].

g. Proof by cases

Break the proof up into smaller cases prove the statement in question is true in each case. These smaller cases must cover every possible case.

2) Explain what the following means. $\forall_{\varepsilon>0}\exists_{N\in\mathbb{Z}_{\geq 0}}(n \geq N \Rightarrow |a_n| < \varepsilon)$ (15 points)

This should look something like: For all ____, there is a ____ such that ___ implies ____

For all $\varepsilon > 0$ there is an $N \in \mathbb{Z}_{\geq 0}$ such that $n \geq N$ implies $|a_n| < \varepsilon$.

OR

For all positive epsilon there is a nonnegative integer N such that if n is at least N, then the absolute value of a_n is less than epsilon.

3) Write the following statement in mathematical notation: "There is a number whose square is smaller than any real number" (15 points)

$$\exists_{x\in\mathbb{R}}\forall_{y\in\mathbb{R}}(x^2 < y)$$

(Note that is a false statement)

4) Let $f_n(x) = x^n$. Show that for all $z > 0$, there is an $n \in \mathbb{R}$ such that $f_n(2) < z$. (100 points)

First let's see what this is saying:

$$\forall_{z>0} \exists_{n\in\mathbb{R}} (f_n(2) < z)$$

That is,

$$\forall_{z>0} \exists_{n\in\mathbb{R}} (2^n < z)$$

Let z be a positive real number. Then choose $n = \frac{\ln(z)}{\ln(3)}$. Then we get:

$$f_n(2) = 2^n = 2^{\frac{\ln(z)}{\ln(3)}} < 2^{\frac{\ln(z)}{\ln(2)}} = 2^{\log_2(z)} = z$$

Thus $f_n(2) < z$.