Codename __Solutions____ Transitions, Test 1

(Do not put your name on the test; write your name and codename on the code sheet)

Note that the solutions were created at real time during class. Let me know about any typos or things that need clarification.

1) For the function f, below, show that no two input values give the same output value.

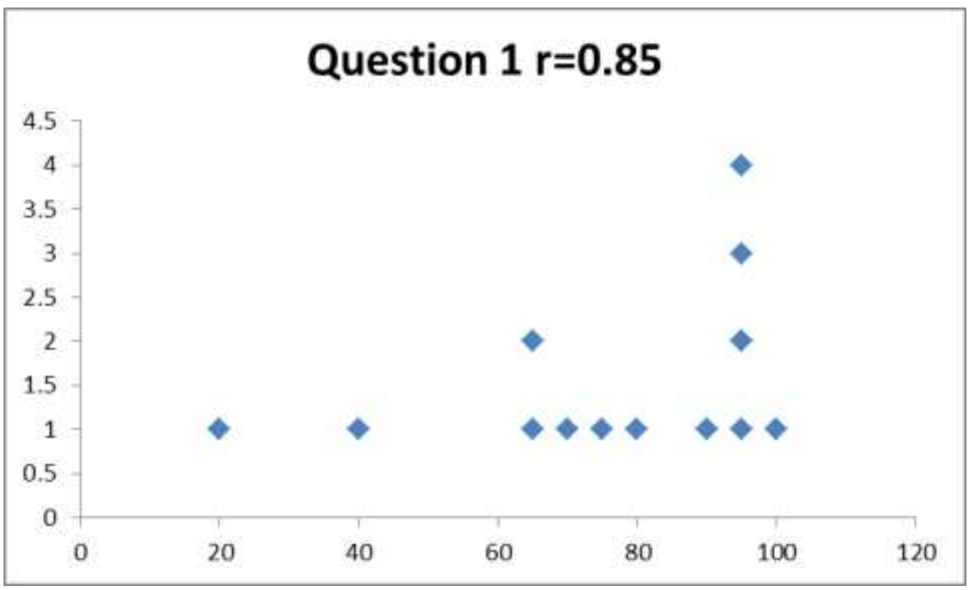
$$f: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto 3x + 4$$

First note that this problem is asking us to prove that f is one-to-one.

To this assume $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \mathbb{R}$.

$$\therefore 3x_1 + 4 = 3x_2 + 4$$
$$\therefore 3x_1 = 3x_2$$
$$\therefore x_1 = x_2$$

Thus we see that $x_1 = x_2$, and so f is injective.



2) Let A and B be sets. Show that $A \cap B = A$ implies $A \subseteq B$.

Assume $A \cap B = A$.
Assume $x \in A$

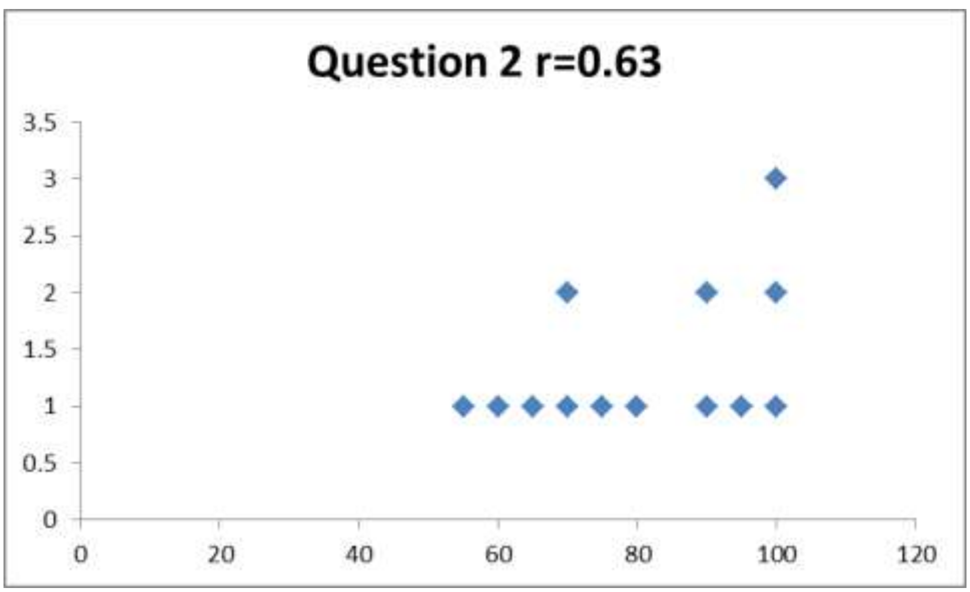
Because $x \in A$, from the assumption we know that $x \in A \cap B$.

Thus $x \in B$ because of $x \in A \cap B$.

We have shown that $x \in B$ whenever $x \in A$.

Therefore $A \subseteq B$ and so $A \cap B = A$ implies $A \subseteq B$.

Alternative proof:
Assume $A \cap B = A$. This means that both $A \cap B \subseteq A$ and $A \subseteq A \cap B$. Using the fact that $A \subseteq A \cap B$, and $A \cap B \subseteq B$, we have that $A \subseteq B$ by the transitivity of the subset operator.



Codename ____ Transitions, Sheet 2

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3) Show that:

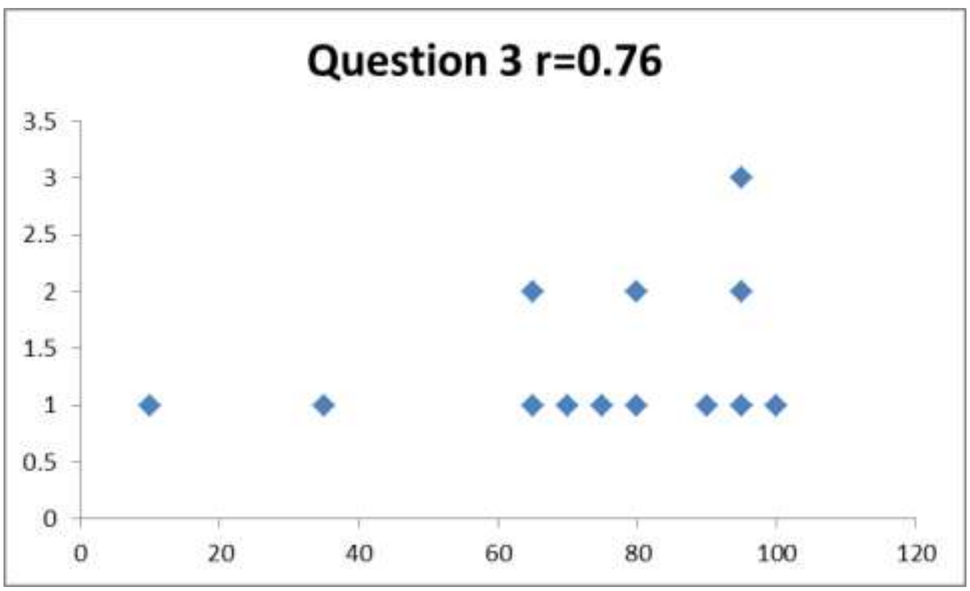
$$\forall_{a\in\mathbb{Z}}\forall_{b\in\mathbb{Q}}\left(\left(a+b\sqrt{5}\right)\cdot\left(a-b\sqrt{5}\right)\in\mathbb{Q}\right)$$

Let $a\in\mathbb{Z}$ and $b\in\mathbb{Q}$. Then let's do some algebra:

$$\left(a+b\sqrt{5}\right)\cdot\left(a-b\sqrt{5}\right)=a^2+ab\sqrt{5}-ab\sqrt{5}-b^2\left(\sqrt{5}\right)^2=a^2-5b^2$$

Note that a and b are both rational, so a^2-5b^2 is also certainly rational.

Therefore $\forall_{a\in\mathbb{Z}}\forall_{b\in\mathbb{Q}}\left(\left(a+b\sqrt{5}\right)\cdot\left(a-b\sqrt{5}\right)\in\mathbb{Q}\right)$.



Codename ____ Transitions, Sheet 3

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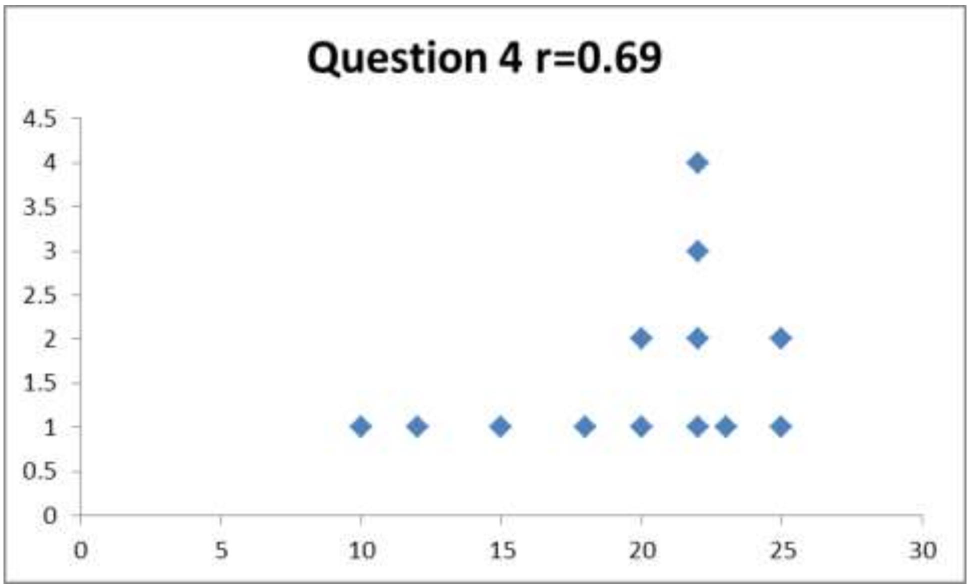
4) Translate this sentence into mathematical symbolism.

"There is a real number whose negation is larger than some complex number"

$$\exists_{x\in\mathbb{R}}\exists_{y\in\mathbb{C}}(-x > y)$$



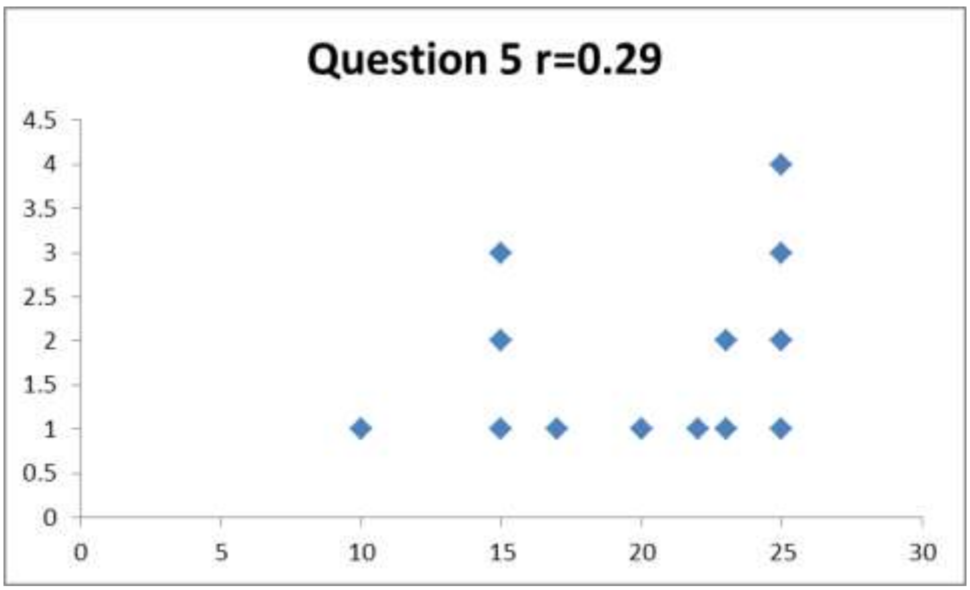
5) Translate this mathematical expression into a sentence.

$$\forall_{x\in\mathbb{C}}\exists_{y\in\mathbb{Z}_{>0}}(|x| < y)$$

For all complex numbers, its absolute value is smaller than some positive integer.

Or

For all complex numbers, there is a positive integer that is larger than the absolute value of the complex number.



6) Prove that:

$$\bigcup_{n=0}^{\infty}\{n,-n\} = \mathbb{Z}$$

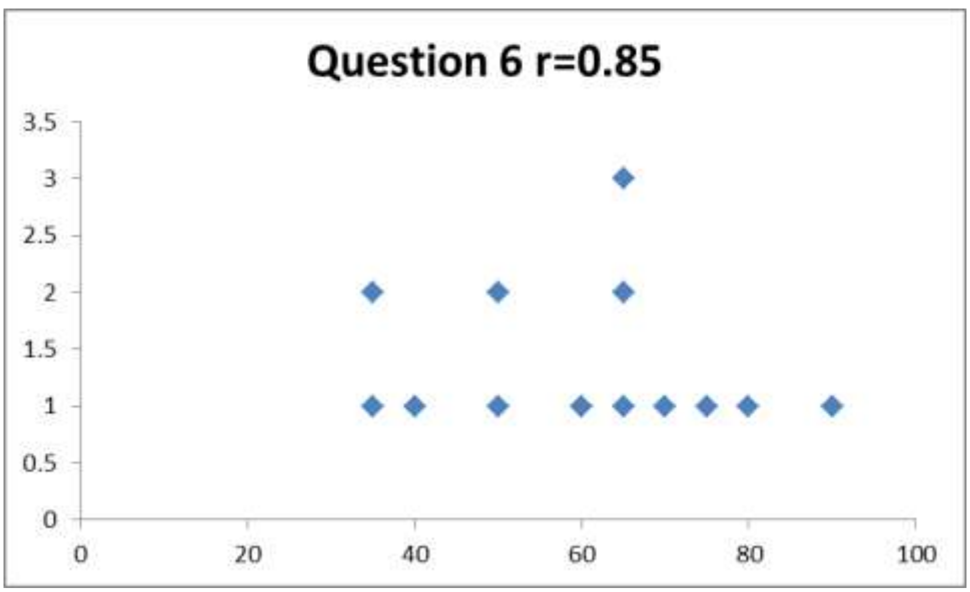
We'll use the typical double subset argument.

Assume $x \in \bigcup_{n=0}^{\infty}\{n,-n\}$. Then there is some index $j \in \mathbb{Z}_{\geq 0}$ such that $x \in \{j,-j\}$. Both j and $-j$ are integers, so either way $x \in \mathbb{Z}$. (x has to be either j or $-j$).

On the other assume $x \in \mathbb{Z}$. Then $x \in \{x,-x\}$. Hence $x \in \bigcup_{n=0}^{\infty}\{n,-n\}$.

Therefore

$$\bigcup_{n=0}^{\infty}\{n,-n\} = \mathbb{Z}.$$



Codename ____ Transitions, Sheet 4

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7) Let A, B and C be sets such that $A \neq \emptyset$. Assume that $A \times B = A \times C$. Prove that $B = C$.

Let's do another double subset argument. We seem to do a lot of these…

Assume $x \in B$. Because $A \neq \emptyset$, there is some $a \in A$. Thus $(a, x) \in A \times B$. Hence $(a, x) \in A \times C$, and so $x \in C$.

Similarly if $x \in C$, we find that $x \in B$. Hence $B = C$.