Codename ____Solutions____ Transitions, Test 2

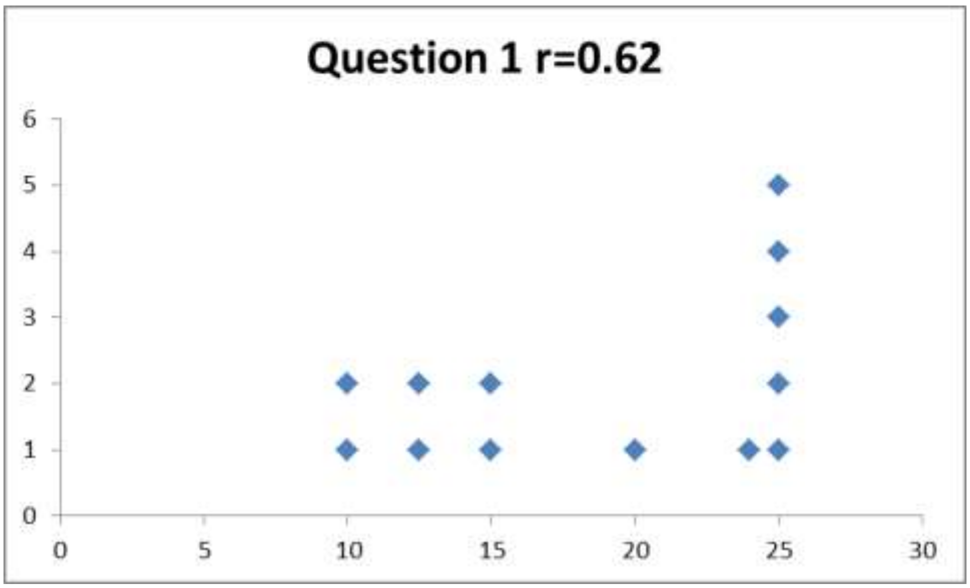
(Do not put your name on the test; write your name and codename on the code sheet)

Define $\mathbb{Z}_5 = \{0,1,2,3,4\}$ to be the integers mod 5.

1) What does it mean for a function $f: \mathbb{Z}_5 \to \mathbb{Z}_5$ to be one-to-one? (25 points)

Every output comes from at most one input:

If $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \mathbb{Z}_5$, then $x_1 = x_2$.



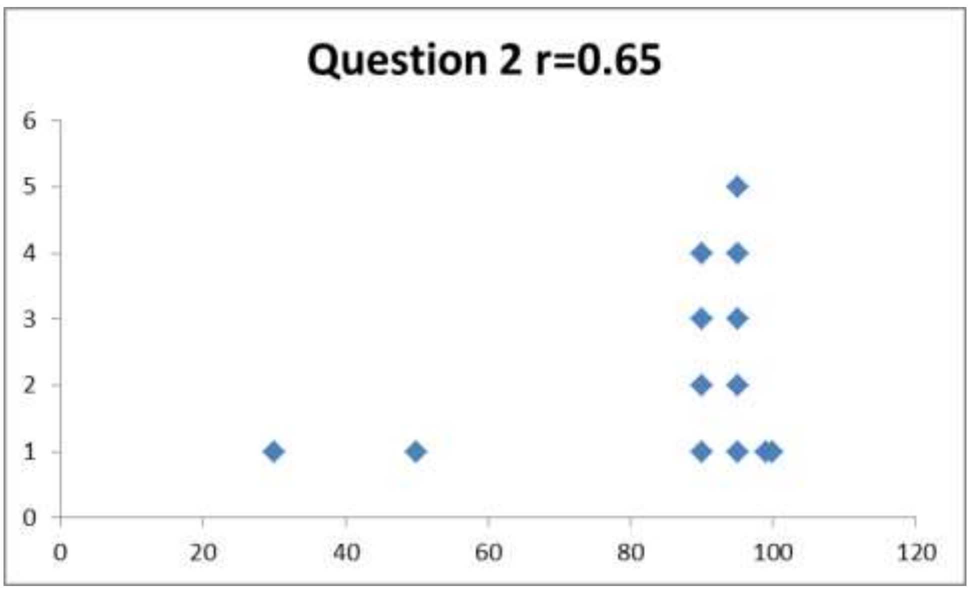
2) Prove that the function below is one-to-one. (100 points. Be sure not to skip steps)

$$f: \mathbb{Z}_5 \to \mathbb{Z}_5$$
$$x \mapsto 2x$$

Assume that $f(x_1) = f(x_2)$ where $x_1, x_2 \in \mathbb{Z}_5$. Thus $2x_1 = 2x_2$. Then we'll multiply this equation by $2^{-1} = 3$:

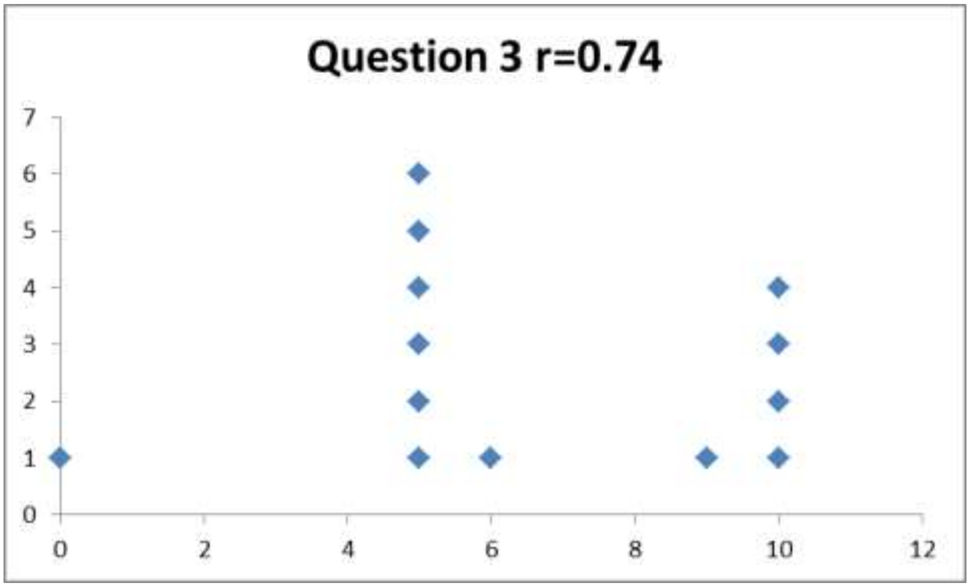
$$3 \cdot 2x_1 = 3 \cdot 2x_2$$
$$\therefore x_1 = x_2$$

Therefore f is injective.



3) What is $\mathbb{R}^2$? Express it as a set. (10 points)

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) | x, y \in \mathbb{R}\}$$



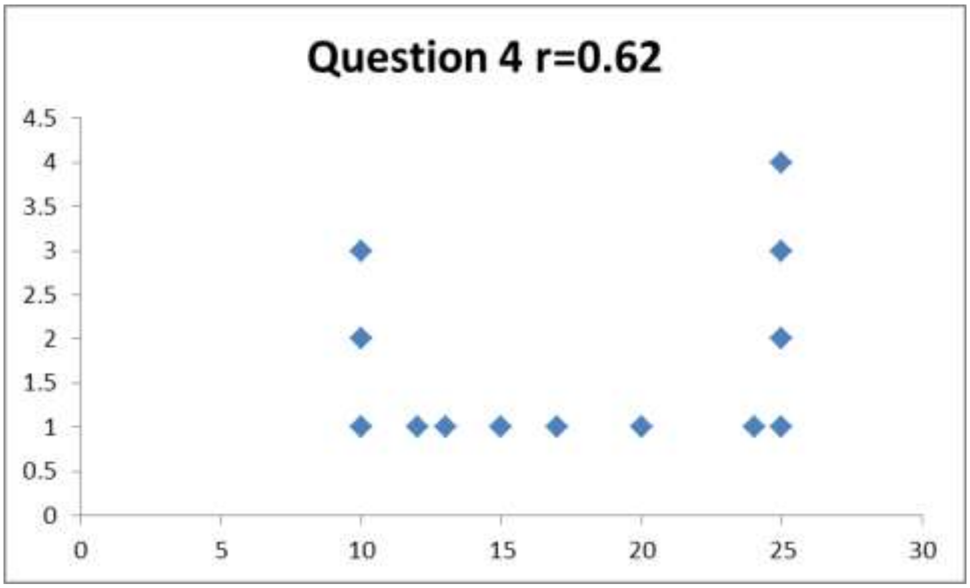
4) What does it mean for a function $f: \mathbb{R}^2 \to \mathbb{R}^2$ to be onto? (25 points)

Every output comes from some input:

For every $y \in \mathbb{R}^2$ there is some $x \in \mathbb{R}^2$ such that $f(x) = y$.

***Note here that x and y are ordered pairs. You could make use of this if you like:

For every $(c, d) \in \mathbb{R}^2$ there are some $(a, b) \in \mathbb{R}^2$ such that $f(a, b) = (c, d)$



5) Prove that the function below is onto. (100 points)

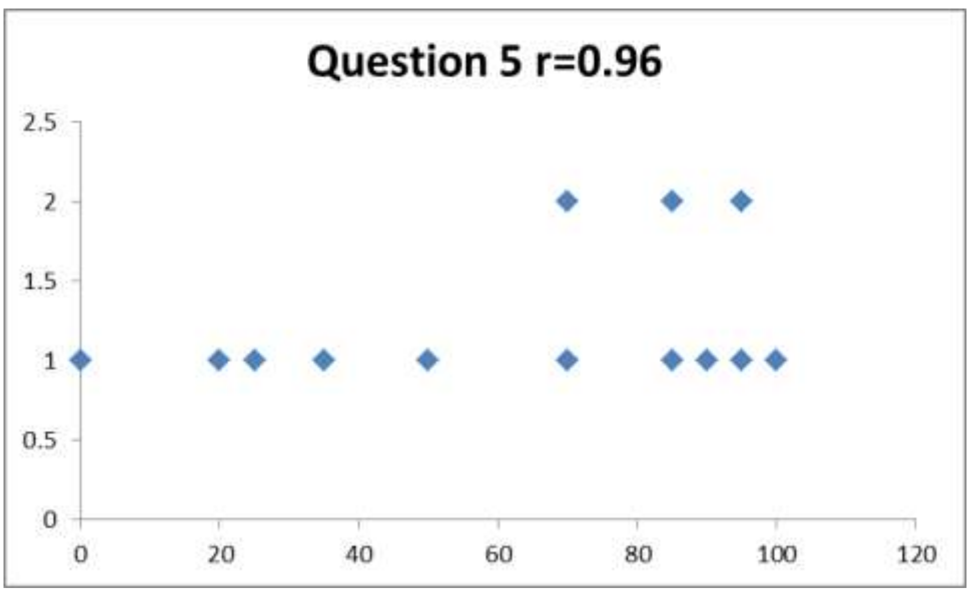
$$f: \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto (x, 2y)$$

Let $(c, d) \in \mathbb{R}^2$. We must find $(a, b) \in \mathbb{R}^2$ such that $f(a, b) = (c, d)$.

Choose $a = c$ and $b = \frac{d}{2}$, then we get:

$$f(a, b) = (a, 2b) = \left(c, 2\frac{d}{2}\right) = (c, d).$$

Therefore f is onto.



Codename ____ Transitions, Sheet 2

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6) Prove the following equality holds true for all $n \in \mathbb{Z}_{\geq 0}$. (100 points)

$$\sum_{i=0}^{n} (2i+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

Base case: This is when $n = 0$.

$$\sum_{i=0}^{0} (2i+1)^2 = (0+1)^2 = 1 = \frac{3}{3} = \frac{1 \cdot 1 \cdot 3}{3}$$

Assume $\sum_{i=0}^{k} (2i+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$ where k is some arbitrary index, that is: $k \in \mathbb{Z}_{\geq 0}$.

Next we prove the $k+1^{th}$ case:

$$\sum_{i=0}^{k+1} (2i+1)^2 = \sum_{i=0}^{k} (2i+1)^2 + (2(k+1)+1)^2$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3} + (2(k+1)+1)^2$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3} + \frac{3(2k+3)^2}{3}$$

$$= \frac{(2k+3)}{3}[(k+1)(2k+1) + 3(2k+3)]$$

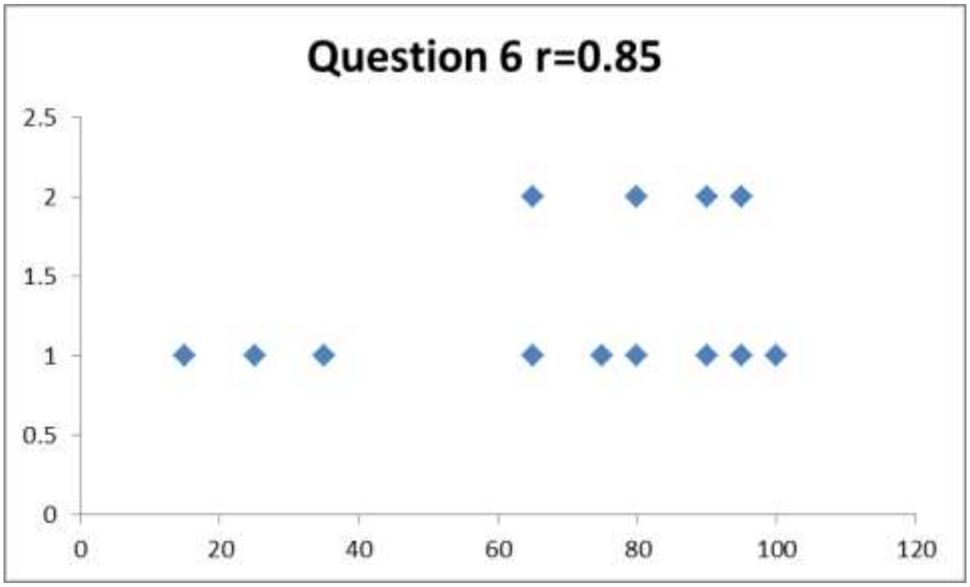
$$= \frac{(2k+3)}{3}[2k^2 + 3k + 1 + 6k + 9]$$

$$= \frac{(2k+3)}{3}[2k^2 + 9k + 10]$$

$$= \frac{(2k+3)(k+2)(2k+5)}{3}$$

$$= \frac{(k+2)(2k+3)(2k+5)}{3}$$

Therefore by induction, $\sum_{i=0}^{n} (2i+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$ for all $n \in \mathbb{Z}_{\geq 0}$.


Question 6 r=0.85
2.5
2
1.5
1
0.5
0
0
20
40
60
80
100
120

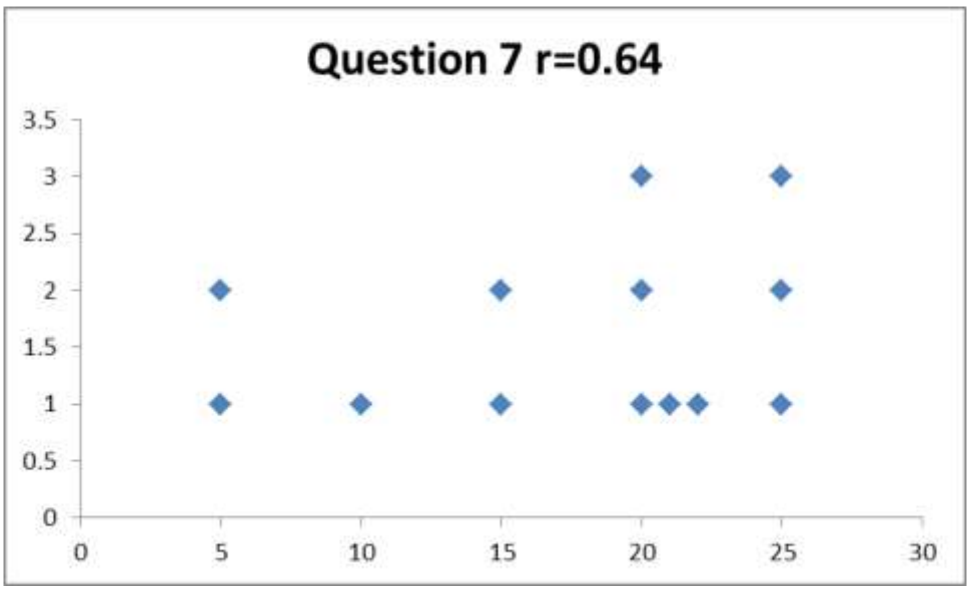
A set S is called well-ordered if every subset of S has a smallest element. That is, every single subset of S has a smallest element.

7) Let S be a well-ordered set. Use the fact that S is well-ordered to construct a partial ordering on S. (50 points)

For any two elements $a, b \in S$, we define a relation via the following:

$$aRb \text{ iff } a \text{ is the smallest element in the set } \{a, b\}$$

We'll call this relation $<_S$ and write $a <_S b$.



8) In fact the relation you constructed is a partial order relation. We won't prove all of this though, just part of it. Prove that your relation is antisymmetric (100 points)

First recall that our relation was given for all elements $a, b \in S$ by:
aRb iff in the set $\{a, b\}$ a is the smallest element according to the well ordering.

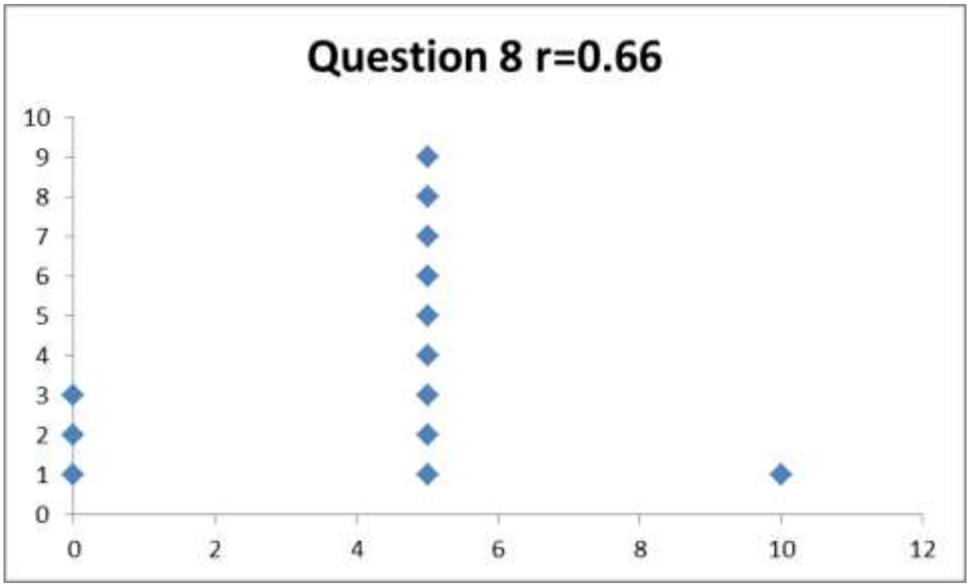
We'll show that the relation $\leq_S$ is antisymmetric by using the contrapositive.

Assume $a \neq b$ where $a, b \in S$. That is, we have two distinct elements of S. Thus by the well ordering, the set $\{a, b\}$ has a smallest element. Without loss of generality, make the smallest element of this set as a.

Thus by the definition of our relation, $a \leq_S b$. Hence $b \leq_S a$ or $a \leq_S b$ (really just the second one, but we need the disjunction in the implication), and so $\leq_S$ is antisymmetric.

Also note that we used the fact that $\sim(c <_S d) \Rightarrow c \geq_S d$ which requires totality, so in the complete proof we must show that $\leq_S$ is total first.

This question was not counted, as nobody came up with the correct relation in the previous question. In some cases a few extra credit points were awarded for something that showed the correct structure.


Question 8 r=0.66
10
9
8
7
6
5
4
3
2
1
0
0
2
4
6
8
10
12

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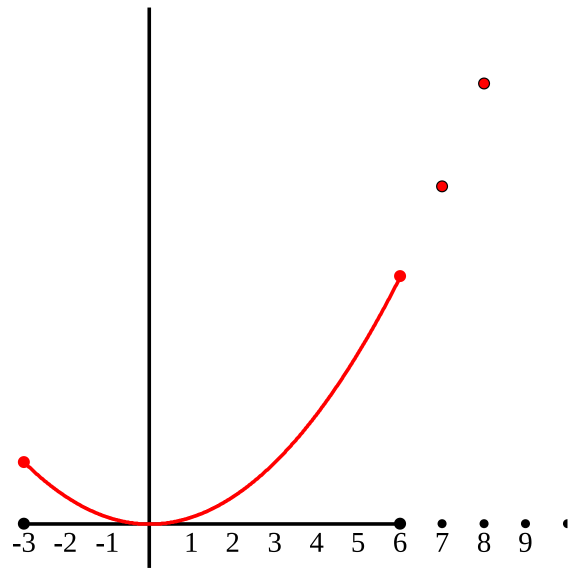
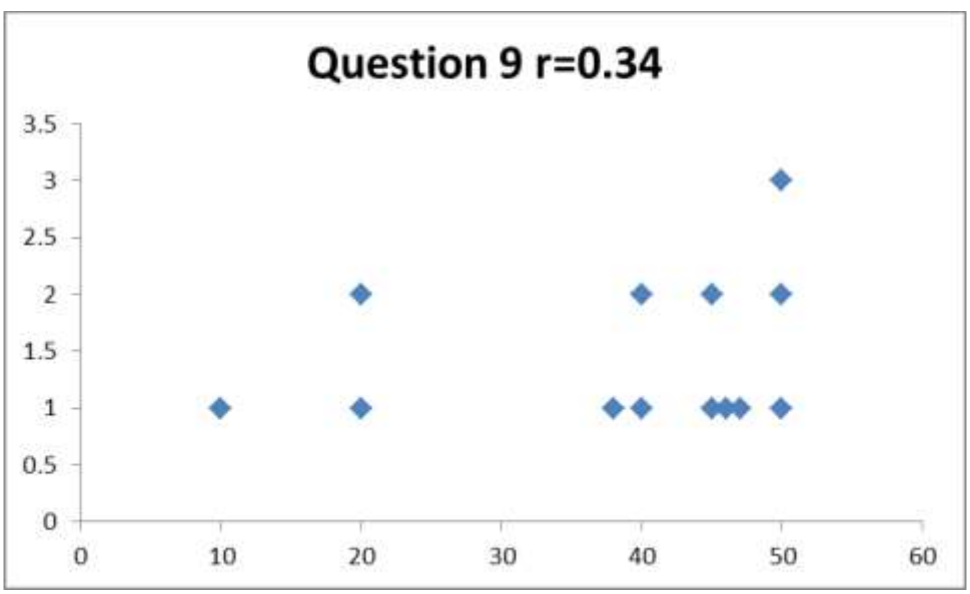
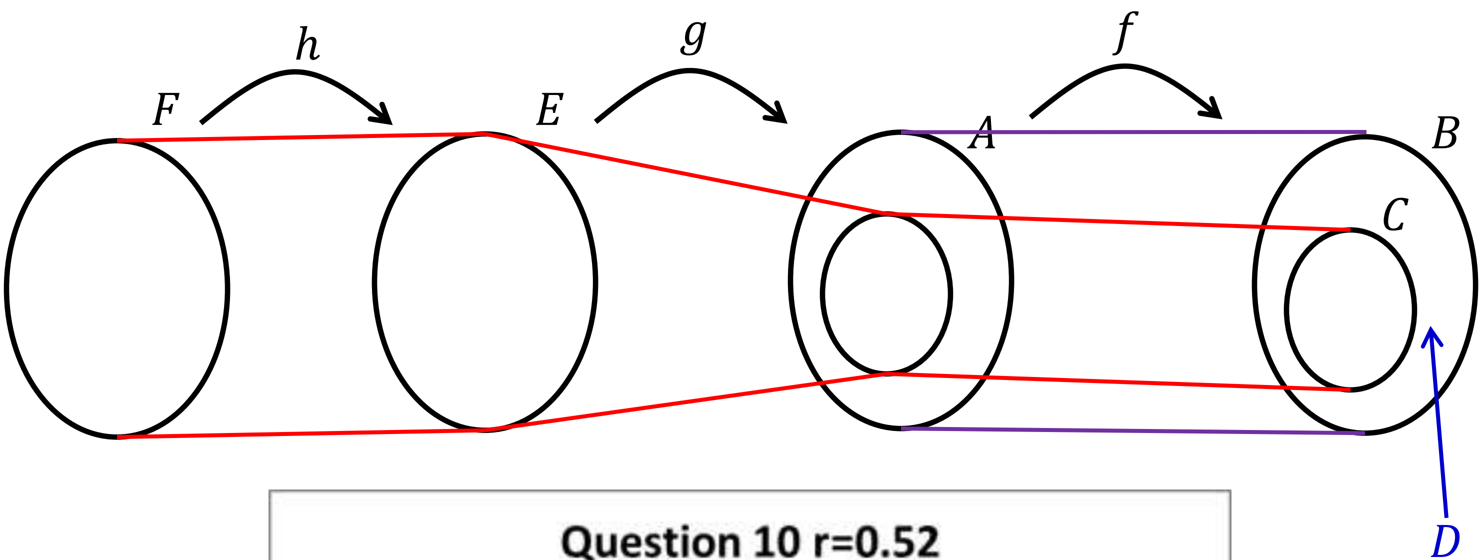
9) Sketch a graph the function below. (50 points)

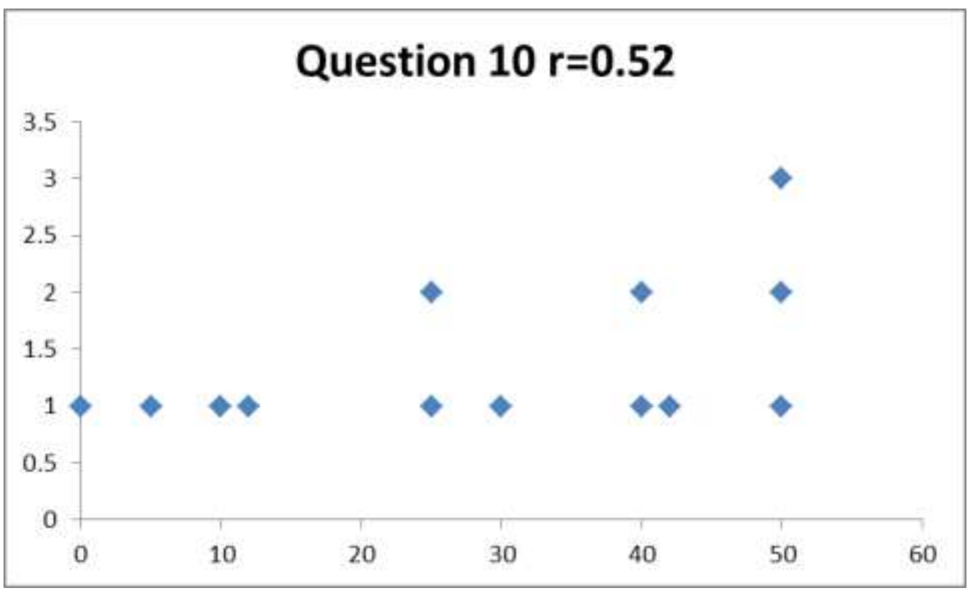
$$f\colon [-3,6] \cup \mathbb{Z}_{\geq 7} \to \mathbb{R}$$
$$x \mapsto x^2$$





10) What is the domain and range of $f \circ g \circ h$? Some properties of f, g, and h are given below. (50 points)
f maps from A onto B, such that the range of g maps to C and the rest of A maps to D.
g maps from E into A.
h maps from F onto E.





11) Complete ONE of the problems below. (100 points)

A) Define relation R on $\mathbb{R}^2$ by identifying points that are the same distance from the origin with each other. Prove that R is an equivalence relation.

B) Define a relation S on the set of all monomials in variables x and y via $x^a y^b S x^c y^d$ iff $a = c$ and $b \leq d$. Prove that S is a partial order relation.