Codename ____ Transitions, Test 1

(Do not put your name on the test; write your name and codename on the code sheet)

An abundant number is a number whose proper factors add up to more than the number itself. For instance, 12 is abundant, but 13 is not.

Consider the statement "There is an abundant number".

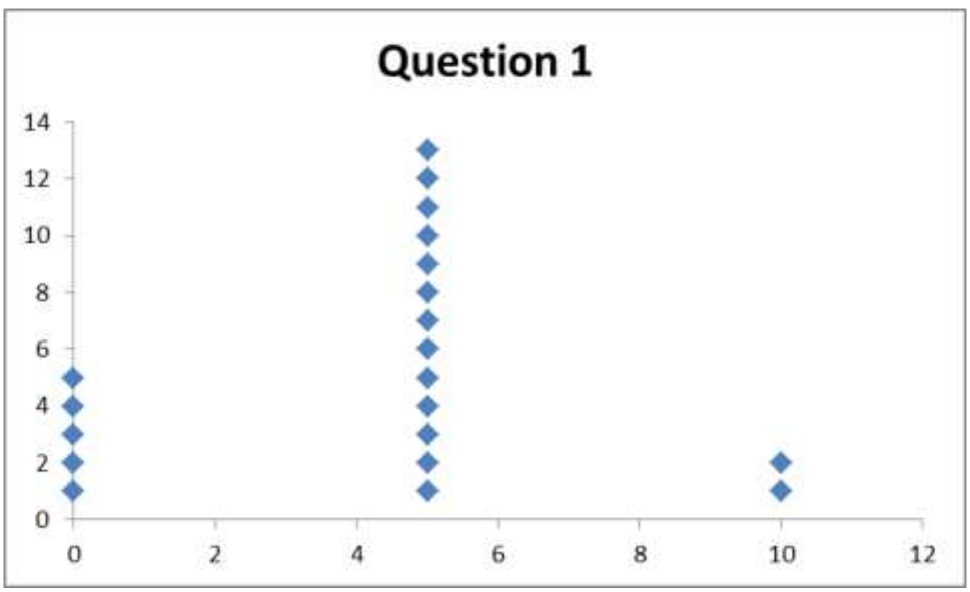
1) Write this statement into mathematical symbolism.

Full credit (5 points) was given for a simple expression of the existential such as:

$$\exists_{\text{number } n}(n \text{ is abundant})$$

5 bonus points were also awarded for expressing what an abundant number is, such as:

$$\exists_{n\in\mathbb{Z}}\left(\sum_{\substack{x|n \\ x\neq n}} x > n\right)$$



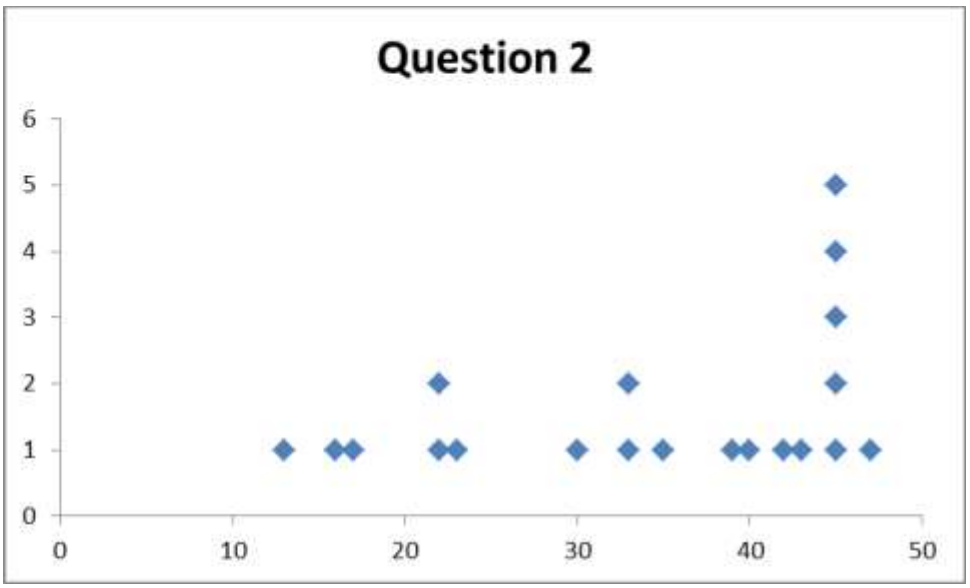
2) Prove this statement

First note that the problem statement tells us that 12 is abundant, so we can use that in the proof:

As given, 12 is an abundant number. Therefore there is an abundant number. $\square$

Or if you work out those details:

Note that $1, 2, 3, 4$, and 6 all divide 12, and that $1 + 2 + 3 + 4 + 6 \geq 12$. Hence 12 is abundant, and so there is an abundant number. $\square$



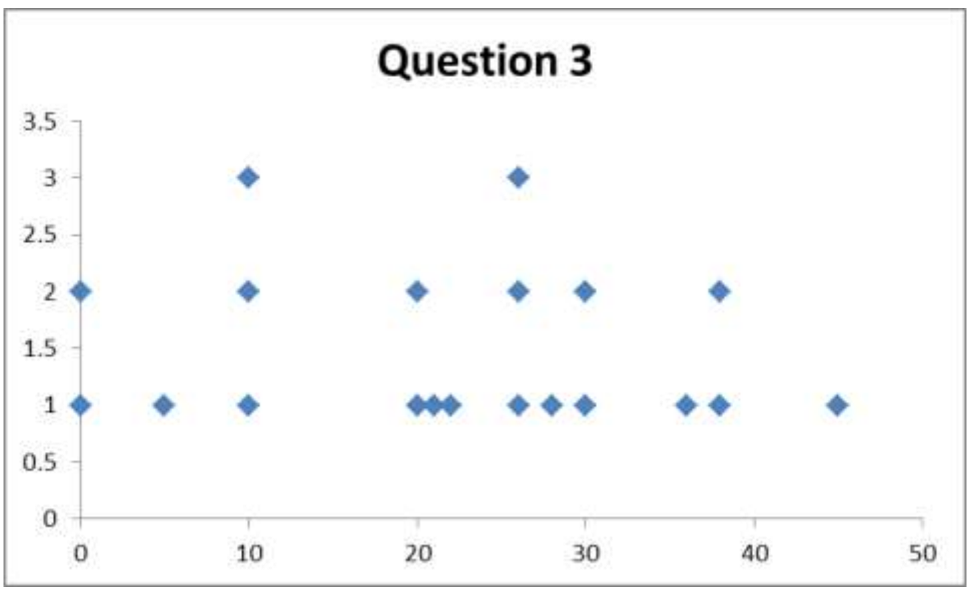
3) Let P, Q, and R be statements. Prove that $[(\sim Q \Rightarrow P) \wedge (Q \Rightarrow R) \wedge (\sim R)] \Rightarrow P$

This can be done either as a truth table or reasoning. The truth table is:

P	Q	R	$\sim Q$	$\sim R$	$\sim Q \Rightarrow P$	$Q \Rightarrow R$	$[(\sim Q \Rightarrow P) \wedge (Q \Rightarrow R) \wedge (\sim R)]$	$[(\sim Q \Rightarrow P) \wedge (Q \Rightarrow R) \wedge (\sim R)] \Rightarrow P$
T	T	T	F	F	T	T	F	T
T	T	F	F	T	T	F	F	T
T	F	T	T	F	T	T	F	T
T	F	F	T	T	T	T	T	T
F	T	T	F	F	T	T	F	T
F	T	F	F	T	T	F	F	T
F	F	T	T	F	F	T	F	T
F	F	F	T	T	F	T	F	T

Or:

Let P, Q, and R be statements. Assume $[(\sim Q \Rightarrow P) \wedge (Q \Rightarrow R) \wedge (\sim R)]$. In particular, $Q \Rightarrow R$, which via the contrapositive is $\sim R \Rightarrow \sim Q$. Combine this with $\sim R$ and we know $\sim Q$. Then combine $\sim Q$ with $\sim Q \Rightarrow P$ which also comes from the assumption and we have P. Therefore $[(\sim Q \Rightarrow P) \wedge (Q \Rightarrow R) \wedge (\sim R)] \Rightarrow P$. $\square$



4) Prove the statement below:

$$\bigcup_{A \in \bigcap_{i \in \mathbb{Z}} \{i\}} A \times A \subseteq \{1, 6, 53\}$$

Don't panic! This looks scary at first. And indeed it is scary, the index of a union is an indexed intersection!! But that's not the only weird thing, the left hand side is a cross product which should consist of ordered pairs, but the right hand side is just a set of three numbers.

How can a set of ordered pairs be a subset of a set of numbers???!

Let's look at that indexing set a little closer:

$$\bigcap_{i \in \mathbb{Z}} \{i\}$$

That's right, that looks something like $\cdots \{-2\} \cap \{-1\} \cap \{0\} \cap \{1\} \cap \{2\} \cdots$ which is clearly empty. Now that we see that, let's make the proof:

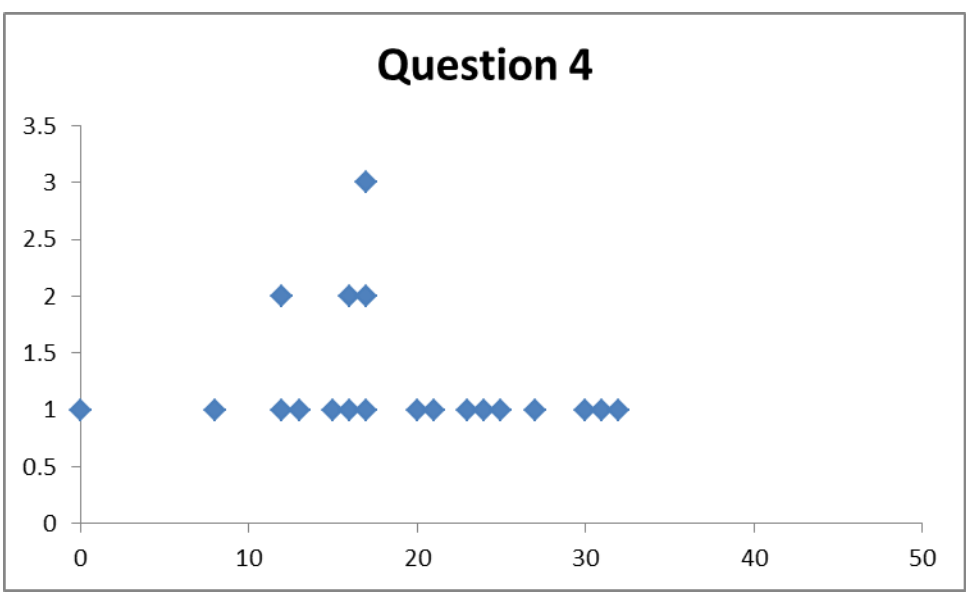
First note that $\bigcap_{i \in \mathbb{Z}} \{i\}$ is an intersection of many sets, in particular the two sets $\{0\}$ and $\{1\}$. As these have an empty intersection, $\bigcap_{i \in \mathbb{Z}} \{i\} = \emptyset$. Thus

$$\bigcup_{A \in \bigcap_{i \in \mathbb{Z}} \{i\}} A \times A = \emptyset$$

Therefore

$$\bigcup_{A \in \bigcap_{i \in \mathbb{Z}} \{i\}} A \times A \subseteq \{1, 6, 53\}$$

□



(Do not put your name on the test; write your name and codename on the code sheet)

5) Prove that $\forall_{x,y\in\mathbb{R}}(x = y = 0 \vee x^2 + y^2 > 0)$

This problem had a typo and read "$\forall_{x,y\in\mathbb{R}}(x = y = 0 \vee x^2 + y^2 = 0)$" which is clearly not true. For instance for $x = 2$ and $y = 7$, neither $x = y = 0$ nor $x^2 + y^2 = 0$ is true. Hence this problem was omitted from grading, although bonus points were given as such: "+5" for noticing that something was sketchy, "$+10$" for noticing that the statement was false.

The proof of the intended statement would go something like this:

Let x and y be arbitrary real numbers. Consider two cases: either $x = y = 0$, or at least one of x, y is nonzero.

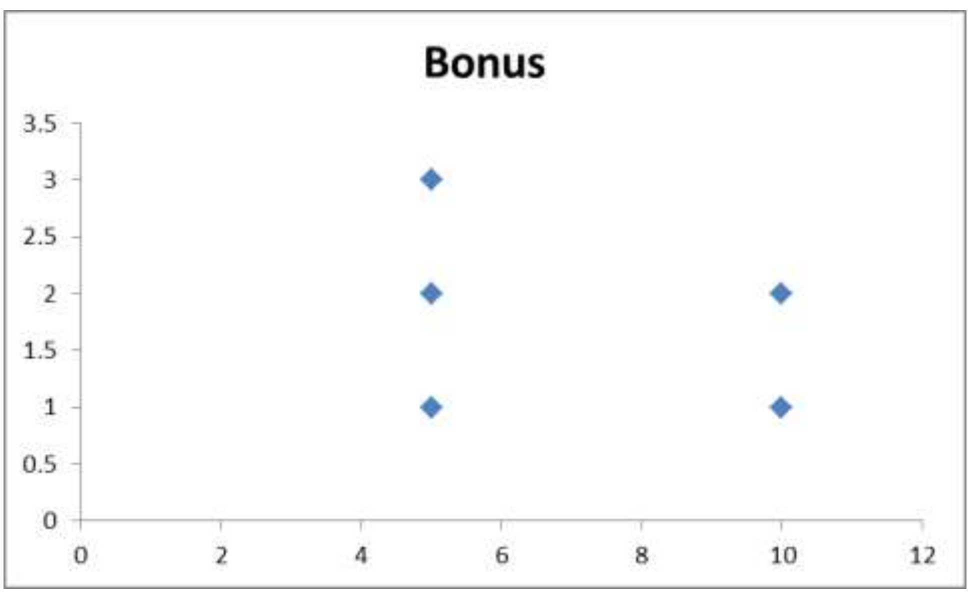
Case 1: If $x = y = 0$, then we are done.

Case 2: If one of x, y is nonzero, assume without loss of generality that $x \neq 0$. We may do this because there are no particular restrictions on one of x, y that is not on the other. Clearly $x^2 + y^2 \geq 0$ because neither x^2 nor y^2 can be negative. Furthermore, because $x \neq 0$, $x^2 > 0$, and so we may conclude:

$$x^2 + y^2 > 0.$$

Hence either $x = y = 0$ or $x^2 + y^2 > 0$.

Therefore $\forall_{x,y\in\mathbb{R}}(x = y = 0 \vee x^2 + y^2 > 0)$. $\square$



6) A real-valued function f is said to be <u>even</u> iff $f(x) = f(-x)$ for all input values x. Write the statement "$f(x) = x^2 + 2$ is even" in explicit mathematical symbolism. Prove the statement.

$\forall_{x\in\mathbb{R}}(x^2 + 2 = (-x)^2 + 2)$

Or

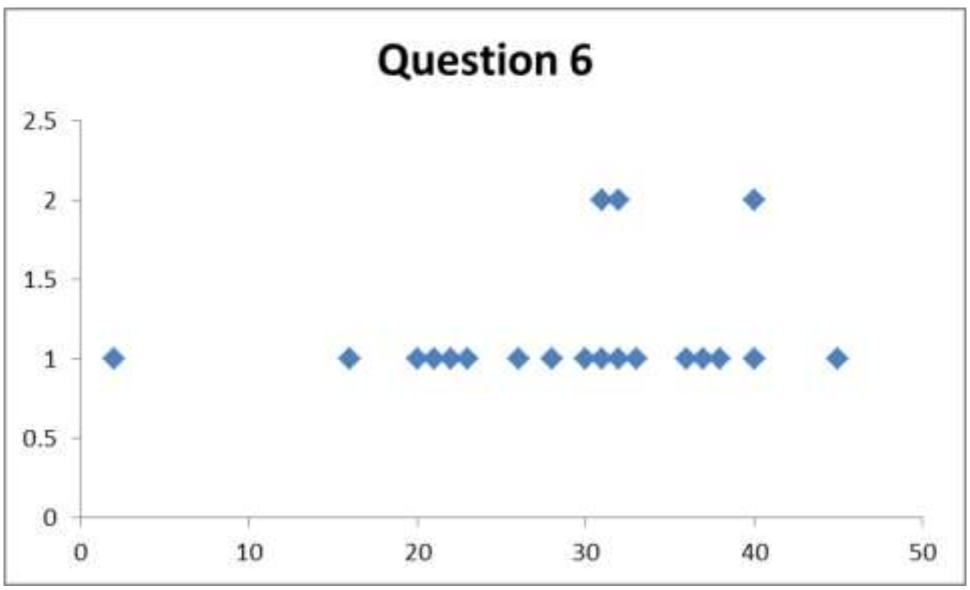
Let $f(x) = x^2 + 2$. $\forall_{x\in\mathbb{R}}\big(f(x) = f(-x)\big)$

To prove this we can do the following.

We will show that $f(x) = x^2 + 2$ is even. Let $x \in \mathbb{R}$. Then:

$$f(x) = x^2 + 2 = (-x)^2 + 2 = f(-x).$$

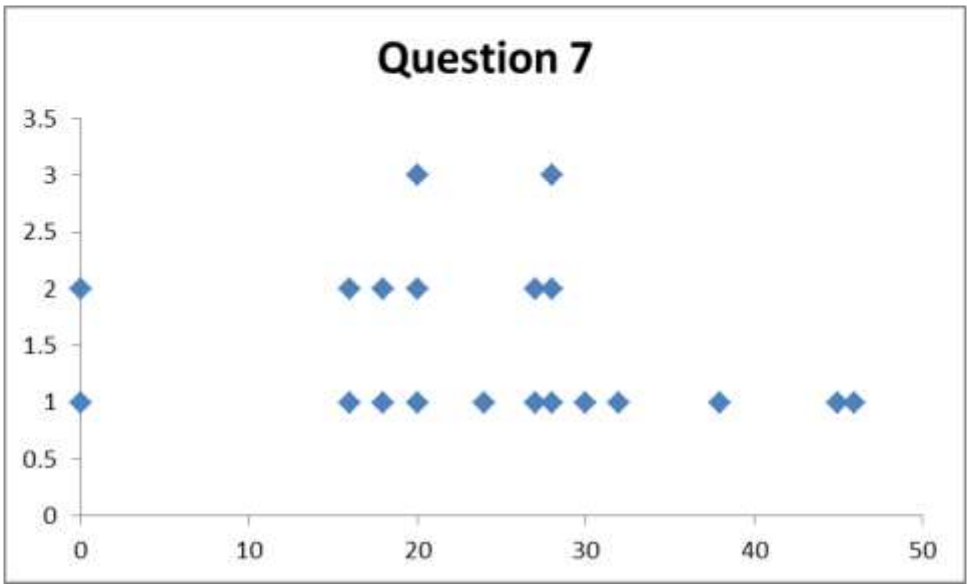
Hence $f(x) = f(-x)$. This holds for all real values of x, and so f is even. $\square$



7) Let A and B be sets. Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

Assume $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$. Thus either $X \in \mathcal{P}(A)$ or $X \in \mathcal{P}(B)$. Without loss of generality assume $X \in \mathcal{P}(A)$. We may do this because A and B are completely symmetrical: there are no restrictions on one that are not on the other.

Now $X \in \mathcal{P}(A)$ means, by definition, that $X \subseteq A$. So also $X \subseteq A \cup B$. This in turn means $X \in \mathcal{P}(A \cup B)$. Therefore $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. $\square$



8) Prove the following statement. $\exists_{x\in\mathbb{R}}\forall_{y\in\mathbb{R}}\exists_{z\in\mathbb{Q}}(xyz \neq 0 \Rightarrow x \in \mathbb{Q})$

Again, don't panic! Indeed that is a lot of quantifiers. But two of them are existential so we'll want to carefully choose x and z. Looking ahead, we see that we'll need to prove $xyz \neq 0 \Rightarrow x \in \mathbb{Q}$. So we know we'll need to assume that $xyz \neq 0$ and show that $x \in \mathbb{Q}$. Ahha! So we need $x \in \mathbb{Q}$. Let's try to prove it now:

Choose $x = 1$.
Let $y \in \mathbb{R}$.
Choose $z = 17.2$.
Assume $xyz \neq 0$.
Then because $x = 1$, indeed $x \in \mathbb{Q}$.
Thus $xyz \neq 0 \Rightarrow x \in \mathbb{Q}$.
Therefore $\exists_{x\in\mathbb{R}}\forall_{y\in\mathbb{R}}\exists_{z\in\mathbb{Q}}(xyz \neq 0 \Rightarrow x \in \mathbb{Q})$. □

Note that because the solution to this problem was so straightforward the 6-tuple grades were simplified to just the 'correctness' component with triple the value.