Codename ____ Transitions, Test 2, 4/22/2014

(Do not put your name on the test; write your name and codename on the code sheet)

1) Prove that for all $n \in \mathbb{Z}_{\geq 1}$:

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$

We shall prove this using induction on n.

Base Case: $n = 1$

$$\sum_{i=1}^{1} i(i+1) = 1 \cdot (1+1) = 2 = \frac{2 \cdot 3}{3} = \frac{1(1+1)(1+2)}{3}$$

Induction Hypothesis: Assume for some $k \in \mathbb{Z}_{\geq 1}$:

$$\sum_{i=1}^{k} i(i+1) = \frac{k(k+1)(k+2)}{3}$$

Induction Step:

$$\begin{aligned}\sum_{i=1}^{k+1} i(i+1) &= \sum_{i=1}^{k} i(i+1) + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} \\ &= \frac{(k+1)(k+2)[k+3]}{3}\end{aligned}$$

Therefore by induction we have proven that for all $n \in \mathbb{Z}_{\geq 1}$:

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}.$$

□

2) Define a relation R on $(\mathbb{Z}-\{0\}) \times (\mathbb{Z}-\{0\})$ via $(a,b)R(c,d)$ iff the following is true:

$$ad = bc$$

a) Give an example of two elements that are related, and another example of two elements that are not related. Ask Dr. Beyerl if they're correct. (This is so that you don't try to make a proof without first getting a feel for the objects you're working with)

$(4,8)R(3,6)$, but $(1,2)\not R(2,3)$

b) Prove that R is an equivalence relation.

Reflexive: We need to prove that for all $A \in (\mathbb{Z}-\{0\}) \times (\mathbb{Z}-\{0\})$, ARA.
Let $A \in (\mathbb{Z}-\{0\}) \times (\mathbb{Z}-\{0\})$.
Without loss of generality we can write A as $A = (a,b)$ where $a, b \in \mathbb{Z}-\{0\}$.
Clearly $ab = ba$, so we can conclude that $(a,b)R(a,b)$.
Therefore ARA.

Symmetric: We need to prove: if $(a,b)R(c,d)$, then $(c,d)R(a,b)$
Assume that $(a,b)R(c,d)$ where $(a,b),(c,d) \in (\mathbb{Z}-\{0\}) \times (\mathbb{Z}-\{0\})$.
$\therefore ad = bc$
$\therefore bc = ad$
$\therefore cb = da$
$\therefore (c,d)R(a,b)$

Transitive: We need to prove: if $(a,b)R(c,d)$ and $(c,d)R(e,f)$, then $(a,b)R(e,f)$.
Assume $(a,b)R(c,d)$ and $(c,d)R(e,f)$ where $(a,b),(c,d),(e,f) \in (\mathbb{Z}-\{0\}) \times (\mathbb{Z}-\{0\})$.
Thus $ad = bc$ and $cf = de$.
Then rearranging both of these to get c and d on the same side of the equation we get:

$$\frac{a}{b} = \frac{c}{d} \quad \text{and} \quad \frac{c}{d} = \frac{e}{f}$$

Thus $\frac{a}{b} = \frac{e}{f}$ and so $af = eb$ which means $(a,b)R(e,f)$.

Therefore R is reflexive, symmetric, and transitive: it is an equivalence relation.

□

(Do not put your name on the test; write your name and codename on the code sheet)

3) Let S be any set. Define a relation on $\mathcal{P}(S)$ via ARB iff $A \cap B \neq \emptyset$.

Prove or disprove each of the following:

a) R is reflexive.

R is not reflexive because $\emptyset \in \mathcal{P}(S)$, so if we take $A = \emptyset, A \cap A = \emptyset$, so $A \not R A$.

b) R is symmetric

Suppose that ARB, then $A \cap B \neq \emptyset$. Thus $B \cap A \neq \emptyset$ and so BRA.

c) R is antisymmetric

R is not necessarily antisymmetric. We shall construct a counterexample.
Let $S = \{1,2,3\}, A = \{1,2\}, B = \{2,3\}$. Then ARB, and BRA, but $A \neq B$.

d) R is transitive

R is not necessarily transitive. We shall construct a specific counterexample.
Let $S = \{1,2,3,4\}, A = \{1,2\}, B = \{2,3\}, C = \{3,4\}$. Then ARB, and BRC, but $A \not R C$.

e) R is total

R is not necessarily total. We shall construct a specific counterexample.
Let $= \{1,2,3\}, A = \{1,2\}, B = \{3\}$. Then neither $A \not R B$, nor is $B \not R A$.

4) Show that the function below is one-to-one.

$$f: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto 9x - 2$$

Let $x, z \in \mathbb{R}$ and assume that $f(x) = f(z)$.

$$\therefore 9x - 2 = 9z - 2$$
$$\therefore 9x = 9z$$
$$\therefore x = z$$

Therefore f is one-to-one.

□

(Do not put your name on the test; write your name and codename on the code sheet)

5) Show that the function below is onto.

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto 9x - 2 \end{aligned}$$

Assume $y \in \mathbb{R}$.

Then choose $x = \frac{y+2}{9}$, and we get:

$$\begin{aligned} f(x) &= 9\left(\frac{y+2}{9}\right) - 2 \\ &= y + 2 - 2 \\ &= y \end{aligned}$$

Therefore f is onto.

□

6) A binomial is a mathematical expression with two terms. In this problem we will work with the variable x and constants. A binomial is an expression of the form:

$$ax + b$$

where $a, b \in \mathbb{R}$.

a) Give 5 examples of expressions that are binomials

$$x + 2$$
$$3x - 5$$
$$14x + \pi$$
$$-2.3x + 7$$
$$23$$

b) Give 5 examples of expressions that are not binomials

$$x^2$$
$$\sqrt{x} + 2$$
$$1/x$$
$$14x + 2 + x^{-1}$$
$$ix + 2$$

Now define an ordering on binomials via: $ax + b \preccurlyeq cx + d$ iff: one of the conditions are satisfied:

1) $a < c$

OR

b) $a = c$ and $b \leq d$

Show that $\preccurlyeq$ is a linear ordering.

Reflexive:

Let $ax + b$ be an arbitrary binomial. Note that $a = a$ and $b \leq b$ so $ax + b \preccurlyeq ax + b$.

Antisymmetric:

Assume that $ax + b \preccurlyeq cx + d$ and $cx + d \preccurlyeq ax + b$ where $ax + b$ and $cx + d$ are binomials.

Note that we cannot have $a < c$ because then $cx + d \preccurlyeq ax + b$ would not be true. Thus $a = c$ and $b \leq d$.

On the other hand we similarly obtain $c = a$ and $d \leq b$.

Therefore $a = c$ and $b = d$, so $ax + b = cx + d$.

Transitive:

Assume that $a_1x + b_1 \preccurlyeq a_2x + b_2$ and $a_2x + b_2 \preccurlyeq a_3x + b_3$ where $a_ix + b_i$ are binomials.

There are several cases:

Case 1: $a_1 < a_2$ and $a_2 < a_3$.

Case 2: $a_1 < a_2$ and $a_2 = a_3$.

Case 3: $a_1 = a_2$ and $a_2 < a_3$.

In each of cases 1-3, $a_1 < a_3$ so $a_1x + b_1 \preccurlyeq a_3x + b_3$.

Case 4: $a_1 = a_2$ and $a_2 = a_3$. In this case we know that also $b_1 \leq b_2$ and $b_2 \leq b_3$. Hence $b_1 \leq b_3$ and so we know that $a_1x + b_1 \preccurlyeq a_3x + b_3$.

Total:

Let $ax + b$ and $cx + d$ be binomials. If $a \neq c$ then either $a < c$ or $c < a$. Each of these gives, respectively, $ax + b \preccurlyeq cx + d$ and $cx + d \preccurlyeq ax + b$.

On the other hand, if $a = c$, then either $b \leq d$ or $d \leq b$ (maybe both). Each of these also gives, respectively, $ax + b \preccurlyeq cx + d$ and $cx + d \preccurlyeq ax + b$.

Hence R is reflexive, antisymmetric, transitive, and total. Therefore R is a linear ordering.

$\square$