Codename ____ Transitions, Test 1

(Do not put your name on the test; write your name and codename on the code sheet)

1) Show that the function f, defined below, is onto. (100 points)

$$f: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto 3x + 9$$

Proof: Let y be arbitrary real number. Choose $x = \frac{y-9}{3}$. Then we get:

$$f(x) = f\left(\frac{y-9}{3}\right) = 3\left(\frac{y-9}{3}\right) + 9 = y - 9 + 9 = y$$

Therefore f is onto.

2) Give an example of a real function that is onto but not one-to-one. (25 points)

There are many examples here. One such example is:

$$f(x) = \begin{cases} x^2, x > 0 \\ -x^2 + 5, x \leq 0 \end{cases}$$

3) Give an example of a real function that is one-to-one but not onto. (25 points)

$$f(x) = 2^x$$

4) In a single English sentence, give a conceptual explanation for the term "injective". (25 points)

Each output value is attained from only one input value.

5) Give a mathematical statement that defines the term "injective". (25 points)

$$\forall_{x_1, x_2 \in \mathbb{R}} (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$$

6) Prove or disprove the statement given below: (100 points)

$$\exists_{x \in \mathbb{Z}}(x \in 2\mathbb{Z} \wedge x > 11)$$

Proof: Choose $x = 12$. Then we see that $x \in 2\mathbb{Z}$ and $x = 12 > 11$.

7) Give an example of a number that is rational. (20 points)

Rational numbers are those that can be written as $\frac{p}{q}$ where $p, q \in \mathbb{Z}$. Examples include:

$$5, \frac{2}{3}$$

8) Give an example of a number that is not rational. (20 points)

Examples include anything that cannot be written as in the previous problem, such as:

$$\pi, e, \sqrt{2}, 7i, \sqrt{7} + 2$$

9) Give an example of a number in $\mathbb{Q}$. (20 points)

$\mathbb{Q}$ is the set of rationals, so this is question 7 again. Examples include $5, \frac{2}{3}$, etc.

10) Give an example of a number in $\mathbb{R} - \mathbb{Q}$. (20 points)

Here we need a real number that is not rational. Examples include:

$$\pi, e, \sqrt{2}, \sqrt{7} + 2$$

11) Give an example of a number in $\mathbb{C} - \mathbb{R}$. (20 points)

Here we need a complex number but not real, such as $7i$.

12) In a single English sentence, what does the statement below mean: (25 points)

$$\exists_{\text{person } p} \forall_{\text{kitten } k} (p \text{ is friends with } k)$$

There is a person that is friends with every kitten.

13) Find the negation of the statement below: (25 points)

$$\forall_{\varepsilon>0} \exists_{N\in\mathbb{Z}} \forall_{n>N} (|a_n - L| < \varepsilon)$$

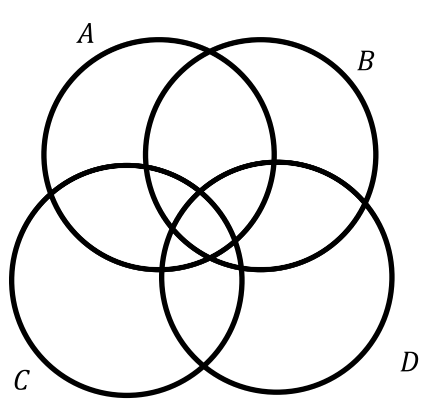
$$\begin{aligned}\sim\forall_{\varepsilon>0} \exists_{N\in\mathbb{Z}} \forall_{n>N} (|a_n - L| < \varepsilon) &\Leftrightarrow \exists_{\varepsilon>0} \forall_{N\in\mathbb{Z}} \exists_{n>N} \big(\sim(|a_n - L| < \varepsilon)\big) \\ &\Leftrightarrow \exists_{\varepsilon>0} \forall_{N\in\mathbb{Z}} \exists_{n>N} (|a_n - L| \geq \varepsilon)\end{aligned}$$

14) Find the following union: (25 points)

$$\bigcup_{n=3}^{\infty} \left(-n, \frac{1}{n}\right)$$

$$\bigcup_{n=3}^{\infty} \left(-n, \frac{1}{n}\right) = \left(-3, \frac{1}{3}\right) \cup \left(-4, \frac{1}{4}\right) \cup \left(-5, \frac{1}{5}\right) \cup \cdots = \left(-\infty, \frac{1}{3}\right)$$

15) On the Venn Diagram below, shade in the set corresponding to $(A \cap B) \cap (C - D)$. (25 points)



Let P be the statement below.

"There is a positive integer M such that for all integers n greater than M, we know that $\frac{1}{n} < 0.26$"

16) Rewrite P in formal mathematical symbols. (50 points)

$$\exists_{M\in\mathbb{Z}_{\geq 1}} \forall_{\substack{n>M \\ n\in\mathbb{Z}}} \left(\frac{1}{n} < 0.26\right)$$

17) Prove that P is true. (100 points)

Choose $M = 5$, then for any integer $n > 5$ we get:

$$\frac{1}{n} < \frac{1}{M} = \frac{1}{5} < 0.26$$

Thus $\frac{1}{n} < 0.26$.

18) Sketch a proof of the fact that that $\sqrt{5}$ is irrational. (100 points)

We'll do this with a proof by contradiction:

Assume that $\sqrt{5} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, where p and q are coprime.

$$\therefore q\sqrt{5} = p$$
$$\therefore 5q^2 = p^2$$
$$\therefore 5|p^2$$
$$\therefore 5|p$$
$$\therefore 25|p^2$$
$$\therefore 25|5q^2$$
$$\therefore 5|q^2$$
$$\therefore 5|q$$

This is a contradiction, we cannot have $5|p$ and $5|q$ because then $\gcd(p, q) \neq 1$.