Codename ____ Transitions, Test 2

(Do not put your name on the test; write your name and codename on the code sheet)

1) Let $S = \overline{1}$ in the "mod 6" relation and $T = \overline{1}$ in the "mod 3" relation. Prove that $S \subseteq T$. (100 points)

$$S = \{\ldots 1, 7, 13, \ldots\} = \{6k + 1 | k \in \mathbb{Z}\}$$
$$T = \{\ldots 1, 4, 7, 10, 13, \ldots\} = \{3k + 1 | k \in \mathbb{Z}\}$$

Proof: Assume $x \in S$. That means we can write $x = 6k + 1$ for some $k \in \mathbb{Z}$. Thus $x = 3(2k) + 1 \in T$. Hence $S \subseteq T$.

2) Find $4 \cdot 5$ mod 13. (30 points)

Method 1:

$4 \cdot 5 \equiv 20 \equiv 7$ mod 13.

Method 2:

$\overline{4} \cdot \overline{5} = \overline{20} = \overline{7}$

3) Solve $4x \equiv 7 \bmod 11$. (40 points)

Method 1: (Brute force)

$$4 \cdot 0 \equiv 0$$
$$4 \cdot 1 \equiv 4$$
$$4 \cdot 2 \equiv 8$$
$$4 \cdot 3 \equiv 12 \equiv 1$$
$$4 \cdot 4 \equiv 16 \equiv 5$$
$$4 \cdot 5 \equiv 20 \equiv 9$$
$$4 \cdot 6 \equiv 24 \equiv 2$$
$$4 \cdot 7 \equiv 28 \equiv 6$$
$$4 \cdot 8 \equiv 32 \equiv 10$$
$$4 \cdot 9 \equiv 36 \equiv 3$$
$$4 \cdot 10 \equiv 40 \equiv 7$$

Hence the answer is $x \equiv 10 \bmod 11$.

Method 2: (Use the Inverse)

Notice that $4 \cdot 3 \equiv 12 \equiv 1 \bmod 11$. Hence:

$$3 \cdot 4x \equiv 3 \cdot 7$$
$$\therefore x \equiv 21 \equiv 10$$

4) Solve $x^2 \equiv 1 \bmod 4$. (30 points)

Because this is a quadratic, and we know nothing about quadratics in mods, all we can do is brute force:

$$0 \cdot 0 \equiv 0$$
$$1 \cdot 1 \equiv 1$$
$$2 \cdot 2 \equiv 4 \equiv 0$$
$$3 \cdot 3 \equiv 9 \equiv 1$$

Hence the answers are $x \equiv 1$ or $x \equiv 3 \bmod 4$.

5) Prove that $3|n^3 + 2n$ for all integers greater than 2. (100 points)

Base case: $3^3 + 2 \cdot 3 = 27 + 6 = 33 = 3 \cdot 11$, hence $3|3^3 + 2 \cdot 3$.

Induction Hypothesis: Assume $3|k^3 + 2k$ for some integer $k > 2$. In particular we may write $k^3 + 2k = 3m$ for some $m \in \mathbb{Z}$

Induction Step:

$$\begin{aligned}(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= (k^3 + 2k) + (3k^2 + 3k + 3) \\ &= 3m + 3(k^2 + k + 1) \\ &= 3(m + k^2 + k + 1)\end{aligned}$$

Hence $3|(k+1)^3 + 2(k+1)$, and therefore by induction, for all integers $n > 2$, $3|n^3 + 2n$.

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6) Prove that the relation, given below, which is defined on the integers is transitive. (100 points)

$$xRy \text{ iff } x|y$$

Proof: Assume xRy and yRz for some integers x, y, z. Thus $x|y$ and $y|z$, meaning that there are some other integers $k_1$and k_2 such that:

$$y = xk_1$$
$$z = yk_2$$

Now plug in $y = xk_1$ into the second equation to get:

$$z = xk_1k_2$$

Thus $x|z$, and so the relation is transitive.

7) Prove that the relation, given below, which is defined on the integers is not transitive. (50 points)

$$xRy \text{ iff } x|2y$$

Choose $x = 8, y = 4$, and $z = 2$. Then we see that $8R4$ because $8|2 \cdot 4$ nad $4R2$ because $4|2 \cdot 2$. However, 8 does not divide 4, so 8 is not related to 2.

8) Let S be an arbitrary set. Find a partition of S. (25 points)

$$\mathcal{P} = \{S\}$$

9) Describe how we can construct an equivalence relation from a partition. (25 points)

Given a partition, we construct an equivalence relation by specifying that elements in the same part of the partition are equivalent.

10) A weak ordering relation is defined as a relation that is reflexive, antisymmetric, and transitive. Let R be the relation on the integers given by xRy iff $x \equiv_2 y$ and $x \leq y$. Sketch a proof to show that R is a weak ordering relation. (120 points)

Reflexive: Let x be some arbitrary integer. Thus $x \equiv_2 x$ and $x \leq x$, so xRx. Hence R is reflexive.

Antisymmetric: Assume x and y are integers such that xRy and yRx. Thus $x \equiv_2 y$, $y \equiv_2 x$, $x \leq y$ and $y \leq x$. Looking at those last two, we see that $x = y$. Hence R is antisymmetric.

Transitive: Assume x, y and z are integers such that xRy and yRz. This tells us four things:

$$x \equiv_2 y$$
$$y \equiv_2 z$$
$$x \leq y$$
$$y \leq z$$

By the transitivity of $\equiv_2$, we see that $x \equiv_2 z$. Also because $x \leq y$ and $y \leq z$ we see that $x \leq z$. Hence R is transitive.

Therefore R is a weak ordering relation.

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11) Prove that for all integers $n \geq 5$ that: (100 points)

$$\prod_{m=1}^{n} \frac{1}{2m} \leq \left(\frac{1}{2^n}\right)^2$$

Base case:

$$\prod_{n=1}^{5} \frac{1}{2m} = \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{6} \cdot \frac{1}{8} \cdot \frac{1}{10} \leq \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{1}{4} = \left(\frac{1}{2^5}\right)^2$$

Induction Hypothesis: Assume the following inequality for some integer $k \geq 5$:

$$\prod_{m=1}^{k} \frac{1}{2m} \leq \left(\frac{1}{2^k}\right)^2$$

Induction step:

$$\begin{aligned}\prod_{m=1}^{k+1} \frac{1}{2m} &= \left(\prod_{m=1}^{k} \frac{1}{2m}\right) \cdot \left(\frac{1}{2(k+1)}\right) \\ &\leq \left(\frac{1}{2^k}\right)^2 \cdot \frac{1}{2} \cdot \frac{1}{k+1} \\ &\leq \left(\frac{1}{2^k}\right)^2 \cdot \frac{1}{2} \cdot \frac{1}{2} \\ &= \left(\frac{1}{2^{k+1}}\right)^2\end{aligned}$$

Therefore by induction $\prod_{m=1}^{n} \frac{1}{2m} \leq \left(\frac{1}{2^n}\right)^2$ for all integers $n \geq 5$.

12) Explain what 3^{-1} means mod 7. (60 points)

This is the multiplicative inverse of 3, That is, the number x such that $3x \equiv 1 \bmod 7$.

13) Find $3^{-1} \bmod 7$. (60 points)

$3 \cdot 5 \equiv 15 \equiv 1 \bmod 7$ so $3^{-1} = 5$.

14) Solve $3x + 2 = 6 \bmod 7$. (60 points)

$$3x + 2 \equiv 6$$
$$\therefore 3x \equiv 4$$
$$\therefore x \equiv 5 \cdot 4$$
$$\therefore x \equiv 20 \equiv 6$$

15) How does the following LaTeX code display? (20 points)

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$x_1^2-x^2_1=y+\int_a^b t              dt$
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$$x_1^2 - x_1^2 = y + \int_a^b tdt$$