Name ____ Test 3, Fall 2019

Part 1: Definitions and Concepts

1) Answer true or false for each of the statements below. (10 point each)

T or F (I) A relation is always a function

T or F (II) A function is always a relation

T or F (III) Whenever we have a union of infinitely many sets, the answer is the universe.

T or F (IV) Whenever we have a union of no sets, the answer is the empty set.

T or F (V) When working mod 15, every integer can be reduced to a number in the range 0 through 14.

2) Give the definition of a <u>relation</u> R from a set A to a set B. Be mathematically precise, vague answers will be given no credit. (50 points)

A relation R from A to B is a subset of the cross product $A \times B$.

3) Formally define the modular arithmetic relation by filing in the box below. (50 points)

$x \equiv y$ mod n if and only if $n|x - y$

4) Given the function below, find $f(3,4)$.

$$f: \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \mapsto x^2 y$$

(50 points)

$f(3,4) = 3^2 \cdot 4 = 9 \cdot 4 = 36$

5) Find all square roots of 1 in the mod 8 universe. (50 points)

$$0^2 = 0$$
$$1^1 = 1$$
$$2^2 = 4$$
$$3^2 = 9 \equiv 1$$
$$4^2 = 16 \equiv 0$$
$$5^2 = 25 \equiv 1$$
$$6^2 = 36 \equiv 4$$
$$7^2 = 49 \equiv 1$$

From the brute force approach above, we see that 1 has four square roots mod 8: $1, 3, 5,$ and 7.

6) Reduce 225 mod 13. (25 points)

$225 = 13 \cdot 17 + 4$

$225 \equiv 4$ mod 13.

7) Solve $x + 20 = 15$ mod 13 (25 points)

$x + 20 \equiv 15$ mod 13
$x \equiv -5$ mod 13
$x \equiv 8$ mod 13

Part 2: Proofs

8) Below is a partial proof of the statement below. Finish the proof.

$$\forall_{j\in I}\left(A_j \subseteq \bigcup_{i\in I} A_i\right)$$

(100 points)

Line	Statement	Reasoning
(1)	Let $j \in I$ be an arbitrary index.	N/A – it's the definition of j.
(2)	Assume $x \in A_j$	N/A – It's an assumption.
(3)	$x \in \bigcup_{i\in I} A_i$	Use the definition of union and the fact that $j \in I$.
(4)	$A_j \subseteq \bigcup_{i\in I} A_i$	Apply the definition of subset to the implication formed by lines 2 and 3.
(3)	$\forall_{j\in I}\left(A_j \subseteq \bigcup_{i\in I} A_i\right)$	Universal generalization on line (4) based on the fact that j was arbitrary.

9) Let R be the relation on $\mathbb{R}$ defined by xRy iff $x^2 + y^2 = 0$.

Prove that R is not reflexive.
(50 points)

Choose $x = 1$. $1^2 + 1^2 = 2 \neq 0$. Thus $1 \not R 1$, so R is not reflexive.

Prove that R is symmetric.
(50 points)

Let $a, b \in \mathbb{R}$ and assume aRb. This means that $a^2 + b^2 = 0$. We can apply the commutative property of addition to get $b^2 + a^2 = 0$, which says that bRa. Thus R is symmetric.

10) Prove the inequality below for all integers $n \geq 3$, using induction.

$$n^2 - 7n + 12 \geq 0$$

(100 points)

Base case ($n = 3$): $3^3 - 7 \cdot 3 + 12 = 0 \geq 0$

Induction hypothesis: Assume $k^2 - 7k + 12 \geq 0$ for some integer $k \geq 3$.

Induction step:

$$\begin{aligned}(k+1)^2 - 7(k+1) + 12 &= k^2 + 2k + 1 - 7k - 7 + 12 \\ &= (k^2 + 7k + 12) + 2k + 1 - 7 \\ &\geq 2k + 1 - 7 \\ &\geq 2 \cdot 3 + 1 - 7 \\ &= 0\end{aligned}$$

Thus $(k+1)^2 - 7(k+1) + 12 \geq 0$.

Therefore for all integers $n \geq 3$, $n^2 - 7n + 12 \geq 0$.