Name ____ Test 3, Spring 2019

Part 1: True or False. Circle the correct answer.
(10 points each)

Let R be the relation on $\mathbb{Z}$ given by xRy iff $(x+y)^2 = x^2 + y^2$

T or F 1) R is reflexive.
T or F 2) R is symmetric.
T or F 3) R is antisymmetric.
T or F 4) R is transitive.
T or F 5) R is an equivalence relation.
T or F 6) R is a partial order relation.

Let S be the relation on the set of real polynomials with variable x given by fSg iff $f(0) \geq g(0)$

T or F 7) S is reflexive.
T or F 8) S is symmetric.
T or F 9) S is antisymmetric.
T or F 10) S is transitive.
T or F 11) S is an equivalence relation.
T or F 12) S is a partial order relation.

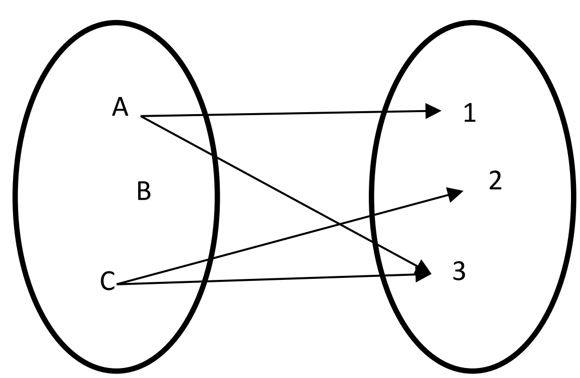
Let A be a set and R a relation on A.

T or F 13) If R is an equivalence relation, it creates a partition of A.
T or F 14) If R is an partial order relation, it generalizes the idea of inequality, "$<$"
T or F 15) If P is a partition of A, then R creates the sets given by P
T or F 16) R cannot be both symmetric and antisymmetric.

Part 2: Definitions and Concepts
(20 points each)

17) For the relation diagram below, write out the relation as a set.

$\{(A,1),(A,3),(C,2),(C,3)\}$



18) Perform the following operations in $\mathbb{Z}_{50}$.

$40 + 25 \equiv 65 \equiv 15$

$15 - 40 \equiv -25 \equiv 25$

19) Find $7 \cdot 8$ mod 50.

$7 \cdot 8 \equiv 56 \equiv 6$

20) Solve the equation below in $\mathbb{Z}_{50}$.

$17x \equiv 1$

$$17x \equiv 51 \equiv 17 \cdot 3$$
$$x = 3$$

21) Find all "square roots of 1" mod 8.

1, 3, 5 and 7

22) Solve the equation below.

$3x + 12 \equiv 34$ mod 20

$$3x + 12 \equiv 14$$
$$3x \equiv 2$$
$$3x \equiv 22$$
$$3x \equiv 42 \equiv 14 \cdot 3$$
$$x \equiv 14$$

Part 3: Proofs

*****NOTE: This test may be a little long. Hence you may skip TWO of the proofs by writing "Skip". However, if you have extra time you can gamble for extra credit: for each attempted proof beyond 2, you will receive a -60 point penalty along with whatever point(s) you earn on that problem.*****

23) Let R be the relation on the set $\{A, B\} \times \mathbb{R}$ given by $(x_1, y_1)R(x_2, y_2)$ iff:

both $x_1 = x_2$ and $y_1 \leq y_2$ or both $x_1 = A$ and $x_2 = B$

It is a fact that R is a partial order relation. Below is part of that proof. Fill in the missing details.
(100 points)

Claim: R is antisymmetric.

Line	Statements	Reasoning
(1)	$(x_1, y_1)R(x_2, y_2)$	Premise
(2)	$(x_2, y_2)R(x_1, y_1)$	Premise

Case 1: $x_1 = x_2$:

Line	Statements	Reasoning
(3)	$y_1 \leq y_2$	_The definition of R applied to line 1_
(4)	$y_2 \leq y_1$	_The definition of R applied to line 2 _
(5)	$y_1 = y_2$	The antisymmetric property of $\mathbb{R}$ applied to lines 3 and 4.

Case 2: $x_1 \neq x_2$:

(6) This case is not possible because either line (1) or line (2) would be false.

Line	Statements	Reasoning
(7)	$(x_1, y_1) = (x_2, y_2)$	Rephrase the conclusion from case (1), while case (2) was impossible.
(8)	R is antisymmetric	The definition of antisymmetric applied to lines (1), (2), and (7)

24) Prove the statement below for all $n \geq 1$

$$\sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$$

(100 points)

Base Case ($n = 1$):

$$\sum_{j=1}^{1} j(j+1) = 1(1+1) = 2 = \frac{2 \cdot 3}{3} = \frac{1(1+1)(1+2)}{3}$$

Assume $\sum_{j=1}^{k} j(j+1) = \frac{k(k+1)(k+2)}{3}$ for some $k \in \mathbb{N}$

$$\sum_{j=1}^{k+1} j(j+1) = (k+1)(k+2) + \sum_{j=1}^{k} j(j+1)$$

$$= (k+1)(k+2) + \frac{k(k+1)(k+2)}{3}$$

$$= (k+1)(k+2)\left(1 + \frac{k}{3}\right)$$

$$= (k+1)(k+2)\left(\frac{3+k}{3}\right)$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

Therefore, for all $n \geq 1$

$$\sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$$

25) Prove the statement below for all $n \geq 6$.

$$n^3 < n!$$

(100 points)

Base case:

$$6^3 = 6 \cdot 6 \cdot 6 = 2 \cdot 3 \cdot 6 \cdot 6 \leq 2 \cdot 3 \cdot 20 \cdot 6 = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 6!$$

Assume $k^3 < k!$ For some $k \in \mathbb{Z}$ with $k \geq 6$.

$$\begin{aligned} (k+1)! = (k+1)k! > (k+1)k^3 \\ = k^4 + k^3 = k^3 + k^4 \\ \geq k^3 + 36k^2 \geq k^3 + 7k^2 \\ = k^3 + 3k^2 + 3k^2 + k^2 \\ \geq k^3 + 3k^2 + 3k + 1 \\ = (k+1)^3 \end{aligned}$$

Therefore, for all $n \geq 6$

$$n^3 < n!$$

26) Prove the statement below for all $n \geq 1$.

$$5|11^n - 6$$

(100 points)

Base case: $11^1 - 6 = 5$, and $5|5$.

Assume $5|11^k - 6$ for some $k \in \mathbb{N}$. This means that $11^k - 6 = 5m$ for some $m \in \mathbb{Z}$.

$$11^{k+1} - 6 = 11 \cdot 11^k - 6 = 11^k - 6 + 10 \cdot 11^k = 5m + 10 \cdot 11^k = 5\left(m + 2 \cdot 11^k\right)$$

Thus $5|11^{k+1} - 6$, and so for all $n \geq 1$ we get

$$5|11^n - 6$$