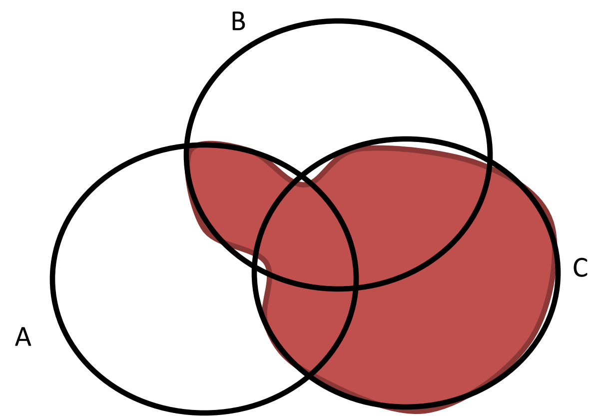
Name ____ Test 2, Spring 2020

Part 1: Definitions and Concepts

1) Illustrate $(A \cap B) \cup C$ on a Venn Diagram.
(50 points)



2) Give the definition of <u>intersection</u>, in the context of sets A and B. Be precise.
(50 points)

$$A \cap B = \{x \in U | x \in A \text{ and } x \in B\}$$

OR

$A \cap B$ is the set of all elements common to both A and B.

3) Let $A = \{1,2,3,4\}$ and $B = \{3,4,5,6\}$. Find $A - B$
(25 points)

$$A - B = \{1,2\}$$

4) Let $A = \{1,2,3,4\}$ and $B = \{3,4,5,6\}$. Find $A \cup B$
(25 points)

$$A \cup B = \{1,2,3,4,5,6\}$$

5) Let $A = [1,4]$ and $B = (3,6)$. Find $A - B$
(25 points)

$$[1,3]$$

6) Let $A = [1,4]$ and $B = (3,6)$. Find $A \cup B$
(25 points)

$$[1,6)$$

7) Find the intersection below.

(50 points)

$$\bigcap_{j=5}^{\infty}\left(1-\frac{1}{j},2-\frac{1}{j}\right)$$

$$\bigcap_{j=5}^{\infty}\left(1-\frac{1}{j},2-\frac{1}{j}\right)=\left(\frac{4}{5},\frac{9}{5}\right)\cap\left(\frac{5}{6},\frac{11}{6}\right)\cap\left(\frac{6}{7},\frac{13}{14}\right)\cap\cdots\left[1,\frac{9}{5}\right)$$

Note that every interval includes 1, but none of the intervals include $\frac{9}{5}$.

8) Answer each of the following as true or false.
(10 points each)

T or F (I) Mathematical induction is used to prove a universally quantified statement.

T or F (II) Mathematical induction requires a base case.

T or F (III) The induction hypothesis in mathematical induction is an assumption.

T or F (IV) Mathematical induction proves a given statement in infinitely many cases.

T or F (V) Mathematical induction is used to prove expressions that are not statements.

Part 2: Proofs

9) Let A, B, and C be sets. Prove that $A \cap (B - C) \subseteq A \cap B$.
(100 points)

Suppose $x \in A \cap (B - C)$. This means $x \in A$ and $x \in B - C$. Because $x \in B - C$, we know $x \in B$. Therefore $x \in A \cap B$.

This question had a lot of very high grades and a lot of very low grades. Don't forget the basics! To prove a subset assume $x \in \text{LHS}$ and prove $x \in \text{RHS}$.
In this case, the proof should start with "Assume $x \in A \cap (B - C)$"

10) Prove the following statement for all natural numbers n:

(100 points)

$$\sum_{j=1}^{n} \frac{j}{(j+1)!} = 1 - \frac{1}{(n+1)!}$$

Let $n \in \mathbb{N}$

Base case: $(n = 1)$

$$\sum_{j=1}^{1} \frac{j}{(j+1)!} = \frac{1}{(1+1)!} = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{(1+1)!}$$

Induction Hypothesis:

Assume $\sum_{j=1}^{k} \frac{j}{(j+1)!} = 1 - \frac{1}{(k+1)!}$ for some natural number k.

Inductive Step:

$$\sum_{j=1}^{k+1} \frac{j}{(j+1)!} = \sum_{j=1}^{k} \frac{j}{(j+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{k+2}{(k+1)!\,(k+2)} + \frac{k+1}{(k+2)!}$$

$$= 1 + \frac{k+1-k-2}{(k+2)!} = 1 + \frac{-1}{(k+2)!} = 1 - \frac{1}{(k+2)!}$$

11) Using Induction, prove ONE of the statements below.
(100 points)

$$n^3 < n! \text{ for all integers } n \geq 7$$

(Hint $7^3 = 343$ and $7! = 5040$)

OR

For all natural numbers n: $8|5^{2n} - 1$

Let $n \geq 7$ be an integer.
Base case: $(n = 7)$

$$7^3 = 343 < 5040 = 7!$$

Induction Hypothesis:
Assume $k^3 < (k+1)!$ for some natural number $k \geq 7$.

Inductive Step:

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1 \leq k! + 3k^2 + 3k + 1 \leq k! + 3k^3 + 3k^3 + k^3 = k! + 7k^3 \leq k! + 7k!$$
$$= 8k! \leq (k+1)k! = (k+1)!$$

Let $n \in \mathbb{N}$
Base case: $(n = 1)$

$$5^{2 \cdot 1} - 1 = 24 = 8 \cdot 3$$

Induction Hypothesis:
Assume $8|5^{2k} - 1$ for some $k \in \mathbb{N}$
$\therefore 5^{2k} - 1 = 8l$ for some $l \in \mathbb{N}$

Inductive Step:

$$5^{2(k+1)} - 1 = 5^{2k} \cdot 5^2 - 1 = 25 \cdot 5^{2k} - 1 = 5^{2k} - 1 + 24 \cdot 5^{2k} = 8l + 8 \cdot 3 \cdot 5^{2k} = 8\left(l + \dot{3}5^{2k}\right)$$
$$\therefore 8|5^{2(k+1)} - 1$$

Therefore $8|5^{2n} - 1$ for all $n \in \mathbb{N}$