Name ____ Test 1, Fall 2021

Grading Note:
Part 3 didn't turn out as I had expected, so in the end I dropped questions 11, 12, 13, 14, and 16. To compute your grade add up the points you earned ignoring those questions and divide by the total number of points attempted. If this didn't result in a higher score, the original raw score was preserved.

Rationale:
I'm not quite sure what happened, but the scores overall on part 3 were significantly lower than I expected. Low scores are not in themselves justification for changing test grades. However, I also did an item analysis on every test question by calculating the correlation between current course grades and the individual question.

Those 5 questions stood out as absolutely terrible questions insomuch as they had a strong *negative* correlation with current course grades:

- r=-0.53,-0.53,-0.76,-0.54,-0.52.

The other questions in part 3 I allowed to stay, as they had a weak but at least positive correlation:

- r=0.17, 0.06, 0.27, 0.09.

The ultimate issue I am taking with questions that have a strong negative correlation with current course grades is that those questions did the opposite of what was intended. People who appear to generally understand the material tended to score poorly on those questions, while people that are doing poorly in the course otherwise tended to score well on those questions. This is an indicator to me that there was something misleading about those questions that made it difficult for students that understand the material to score well, but yet remain easy for students that do not.

It's also interesting to me that none of these questions had a weak negative correlation. It was either positive, or strongly negative. This is yet another indicator that there was something fishy going on in those 5 problems.

In each part, you should skip one problem. If you make an attempt on every problem, clearly cross out the problem you do not want graded. If all are attempted and not crossed out, the last one will not be graded.

Part 1: Basic Knowledge (5 points each, 20 points total)

1) Let U be the universe, x be a variable, and $P(x)$ be an open statement. Define what the notation below means.

$$\forall_{x \in U}\left(P(x)\right)$$

$\forall_{x \in U}\left(P(x)\right)$ is the statement that is true iff $P(a)$ is true for all $a \in U$.

2) Let A and B be sets. Define what the notation below means.

$$A \subseteq B$$

If $x \in A$, then $x \in B$.

3) What is a statement? Give the definition. Be precise.

A sentence of mathematical expression that is either true or false.

4) Let x be an integer. What does it mean for x to be even? Give the definition. Be precise.

$x = 2k$ for some $k \in \mathbb{Z}$

5) Let x be a real number. What does it mean for x to be rational? Give the definition. Be precise.

$x = \frac{a}{b}$ for some integers a, b.

Part 2: Basic Skills and Concepts (5 points each, 20 points total)

6) Simplify the statement form below by finding an equivalent statement with no need for parentheses.

$$\sim(\sim(P \vee Q) \Rightarrow R)$$

$$\sim(\sim(P \vee Q) \Rightarrow R)$$
$$\Leftrightarrow \sim(P \vee Q) \wedge \sim R$$
$$\Leftrightarrow \sim P \wedge \sim Q \wedge \sim R$$

7) Fill in the missing parts of the truth table below for the statement $(P \wedge Q) \Rightarrow R$.

P	Q	R	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

8) Find and simplify the negation of the statement below.

$$\forall_{x \in \mathbb{R}} \exists_{y \in \mathbb{R}} (xy = 0)$$

$$\exists_{x \in \mathbb{R}} \forall_{y \in \mathbb{R}} (xy \neq 0)$$

9) Answer each of the following as True Statement (T), False statement (F), or Not a statement (N).

T F (N) a) 2
(T) F N b) $2 + 2 = 4$
T (F) N c) $\forall_{x \in \mathbb{R}} (x + 2 = 4)$
(T) F N d) $\exists_{x \in \mathbb{R}} (x + 2 = 4)$
(T) F N e) $\exists!_{x \in \mathbb{R}} (x + 2 = 4)$

10) Given $A = \{1,2,3,4,5\}$ and $B = \{3,4,5,6,7\}$ find each of the following.

a) $A \cup B$

$$\{1,2,3,4,5,6,7\}$$

b) $A \cap B$

$$\{3,4,5\}$$

Part 3: Identifying Errors in Proofs (5 points each, 40 points total)

In each problem a formal proof is given, except that the proof is incorrect because it has one or more mistakes. Identify the first mistake by:

- Circling the line number of the mistake
- Circling the type of mistake

Other work or explanation is not required.

(They are listed in order of precedence. So if a line has multiple issues, incomprehensible should be selected before false claim, which should be selected before unjustified claim, which should be selected before incorrect logic)

11) Claim: The sum of two even numbers is even.

1.	Let x and y be even numbers	
2.	$x = 2k$ for some $k \in \mathbb{Z}$	Definition of even
3.	$y = 2k$ for some $k \in \mathbb{Z}$	Definition of even
4.	$2k + 2k$	Plug in x & y
5.	$= 2(k + k)$	Distributive property
6.	$x + y = 2(k + k)$	Transitive property of "$=$"
7.	$x + y$ is even	Definition of even

Type of Mistake: (Incomprehensible) (False claim) (Unjustified claim) (Incorrect logic)

Line 3 with incorrect logic because the variable k is reused. It should be a different letter.

12) Claim: Let a and b be real numbers. $|a - b| = |b - a|$

1.	$\lvert a - b\rvert = \lvert(-1)(b - a)\rvert$	Factor
2.	$= \lvert -1\rvert \cdot \lvert b - a\rvert$	Because $\lvert ab\rvert = \lvert a\rvert \cdot \lvert b\rvert$
3.	$= \lvert b - a\rvert$	Because $\lvert -1\rvert = 1$

Type of Mistake: (Incomprehensible) (False claim) (Unjustified claim) (Incorrect logic)

Line 2 with incorrect logic because a and b are reused. It should be $|xy| = |x| \cdot |y|$ for all $x, y \in \mathbb{R}$

13) Claim: $2x + 3y = 6$ has a solution.

1.	Choose $x = 3$	
2.	Choose $y = 0$	
3.	$2x + 3y = 6$	Premise
4.	$2 \cdot 3 + 3 \cdot 0 = 6$	Plug in x and y
5.	$6 = 6$	Algebra

Type of Mistake: (Incomprehensible) (False claim) (Unjustified claim) (Incorrect logic)

Line 3 is with unjustified claim because it's what we're trying to prove.

14) Claim: If $x > 5$, then $2x > 8$.

1. Choose $x = 6$
2. $2x = 2 \cdot 6 = 12 > 8$ — Plug in and do algebra.

Type of Mistake: (Incomprehensible) (False claim) (Unjustified claim) (Incorrect logic)

Line 1 with incorrect logic, because we're not trying to prove an existential.

15) Claim: Let a, b, and c be integers. If $a|b$ and $b|c$, then $a|c$.

1. $a|b$ — Premise
2. $b|c$ — Premise
3. $\frac{b}{a}$ — Division
4. $\frac{b}{c}$ — Division
5. $\frac{a}{c}$ — Plug line 4 into line 3.
6. $a|c$ — Definition of division.

Type of Mistake: (Incomprehensible) (False claim) (Unjustified claim) (Incorrect logic)

Line 3 with incomprehensible because it's not a statement.

16) Claim: If $2 \nmid x$, then $4 \nmid x$

1. Assume $4|x$ — Assumption for a contrapositive proof
2. $x = 4k$ for some $k \in \mathbb{Z}$ — Definition of division
3. $x = 2(2k)$ — Algebra
4. $2|x$ — Definition of division
5. If $2 \nmid x$, then $4 \nmid x$ — Implication from lines 1 to 4.

Type of Mistake: (Incomprehensible) (False claim) (Unjustified claim) (Incorrect logic)

Line 5 with unjustified claim because the contrapositive implication hasn't been established yet.
I also ended up accepting incorrect logic.

17) Claim: $\Big(\big((Q \vee R) \Rightarrow S\big) \wedge (Q \wedge R)\Big) \Rightarrow S$

1. $Q \cap R$ — Premise
2. $(Q \cup R) \Rightarrow S$ — Premise
3. Q — Simplification (T16) on line 1
4. $Q \cup R$ — Addition (T17) on line 3
5. S — Modus Ponens (T18) on lines 2 and 4.

Type of Mistake: (Incomprehensible) (False claim) (Unjustified claim) (Incorrect logic)

Line 1 with incomprehensible. Notice the incorrect symbol. Intersection and statements don't mix.

18) Claim: $x^2 + y^2 = -8$ is inconsistent.

1. Assume $a^2 + b^2 = -8$ for some $a, b \in \mathbb{R}$ — Assumption for a contradiction proof
2. $a^2 \geq 0$ — Positivity axiom
3. $b^2 \geq 0$ — Positivity axiom
4. $a^2 + b^2 \geq 0$ — Add lines 2 and 3
5. > -8 — Property of numbers
6. $-8 > -8$ — Transitivity property on lines1, 4 and 5
7. $x^2 + y^2 = -8$ — Proof by contradiction on lines 1 because $-8 \ngtr -8$

Type of Mistake: (Incomprehensible) (False claim) (Unjustified claim) (Incorrect logic)

Line 7 with false claim. $x^2 + y^2$ cannot be -8.

19) Claim: If n is a natural number, then $\frac{n}{n+1} > \frac{n}{n+2}$.

1. Assume n is a natural number — Premise
2. Assume $n(n+2) \leq n(n+1)$ — Assumption for the purpose of contradiction
3. $n^2 + 2n \leq n^2 + n$ — Algebra
4. $2n \leq n$ — Algebra
5. $2 \leq 1$ — Algebra
6. $\frac{n}{n+1} > \frac{n}{n+2}$ — Apply proof by contradiction on line 2 because $2 \nleq 1$.

Type of Mistake: (Incomprehensible) (False claim) (Unjustified claim) (Incorrect logic)

Line 2 with incorrect logic, because that's not the assumption you want for proof by contradiction.

But it's not technically incorrect, you could make the proof wrong with that assumption, it just won't be as simple. So we'll also accept:

Line 6 with unjustified claim because you came out with the wrong claim from the proof by contradiction.

Part 4: Proofs (10 points each, 20 points total)

20) Prove that the sum of an integer and a rational number is a rational number.

Let $x \in \mathbb{Z}$ and $y \in \mathbb{Q}$.	
$\therefore y = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$.	Definition of rational
$\therefore x + y = x + \frac{a}{b} = \frac{xb}{b} + \frac{a}{b} = \frac{xb+a}{b}$	Algebra
Note that $xb + a \in \mathbb{Z}$	Property* of integers
$\therefore x + y \in \mathbb{Q}$	Definition of rational

*That property is closure, but we hadn't defined that term

21) Let x and y be integers. Prove that if 4 does not divide $2x + y$, then either 2 does not divide x or 4 does not divide y.

1	Prove that: $4 \nmid 2x + y \Rightarrow (2 \nmid x \lor 4 \nmid y)$	
2	Note that the negation of $(2 \nmid x \lor 4 \nmid y)$ is $2 \mid x$ and $4 \mid y$.	
3		
4	Assume $2 \mid x$ and $4 \mid y$.	Assumption for a contrapositive proof
5	$\therefore x = 2k_1$ for some $k_1 \in \mathbb{Z}$	Definition of divides
6	$\therefore y = 4k_2$ for some $k_2 \in \mathbb{Z}$	Definition of divides
7	$\therefore 2x + y = 2(2k_1) + 4k_2 = 4k_1 + 4k_2 = 4(k_1 + k_2)$	Algebra
8	$\therefore 4 \mid 2x + y$	Definition of divides
9	$\therefore 2 \mid x$ and $4 \mid y$ implies $2 \mid 2x + y$.	Conclusion from line 4 to line 8.
10	$\therefore$: $4 \nmid 2x + y$ implies $2 \nmid x \lor 4 \nmid y$	

22) Prove that if $x > 2$ and $2y > 1$, then $6xy > 5$

Assume $x > 2$ and $2y > 1$.
$6xy = x \cdot 6y > 2 \cdot 6y = 6 \cdot 2y > 6 \cdot 1 = 6 > 5$