Name ____ Test 2, Fall 2021

Part 1: Basic Knowledge (5 points each, 10 points total)

1) Let I be an index set and A_k a set for each $k \in I$. Define what the notation below means.

$$\bigcap_{j \in I} A_j$$

This is the intersection of all A_j. It contains all those elements that are in every single A_j for all $j \in I$.

2) Let $f: A \to B$ be a function. Define what it means for f to be <u>surjective</u>, also known as <u>onto</u>.

Everything in the codomain is covered by something in the domain:

$$\forall_{b \in B} \exists_{a \in A} (f(a) = b)$$

Part 2: Basic Skills and Concepts (5 points each, 20 points total)

3) Answer true or false:

a) $5 \in \{5\}$ T
b) $\{5\} \in \{5\}$ F
c) $5 \subseteq \{5\}$ F
d) $\{5\} \subseteq \{5\}$ T
e) $5 = \{5\}$ F

4) Multiple choice: Which of the following excerpts of LaTeX code will create the symbol below.

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a) \sum_{k=1}^5 k^2
b) $$\sum_{k=1}^5 k^2$$
c) [math]\sum_{k=1}^5 k^2[\math]
d) \start\sum_{k=1}^5 k^2\end
```

$$\sum_{k=1}^{5} k^2$$

5) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 5x + 6$. Answer true or false:

a) f is surjective T
b) f is decreasing F
c) f has an inverse T
d) f is an identity F
e) The domain of f is $5x + 6$ F

6) Let $A = [4,7]$ and $B = (5,12)$. Find:

a) $A \cup B$ $[4,12)$
b) $A \cap B$ $(5,7]$
c) $A - B$ $[4,5]$

Part 4: Proofs (10 points each, 60 points total)

7) Let f be defined below. Prove that f is surjective.

$$f: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto 7x - 12$$

Let $b \in \mathbb{R}$.

Choose $a = \frac{b+12}{7}$

$\therefore f(a) = f\left(\frac{b+12}{7}\right) = 7\left(\frac{b+12}{7}\right) - 12 = b + 12 - 12 = b$

$\therefore f(a) = b$

$\therefore f$ is surjective.

8) Let g be defined below. Prove that g is injective.

$$g: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto 5x + 8$$

Let $a, b \in \mathbb{R}$.

Assume $g(a) = g(b)$.

$\therefore 5a + 8 = 5b + 8$

$\therefore 5a = 5b$

$\therefore a = b$

$\therefore g$ is injective.

9) Let f be defined below. Prove that f has an inverse.

$$f: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto 2x$$

Define $g: \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \frac{x}{2}$.

$(f \circ g)(y) = f(g(y)) = f\left(\frac{y}{2}\right) = 2\frac{y}{2} = y$
$\therefore f \circ g = I_{\mathbb{R}}$
$(g \circ f)(x) = g(f(x)) = g(2x) = \frac{2x}{2} = x$
$\therefore g \circ f = I_{\mathbb{R}}$
$\therefore g = f^{-1}$
$\therefore f$ is invertible.

Alternatively, we have a theorem that says that if f is bijective, it has an inverse. So you could have also proven it is bijective and then used that theorem.

10) Prove theorem T68 on the theorem sheet, using only earlier theorems.

T68 says $(A \subseteq B \land C \subseteq D) \Rightarrow (A \cap C \subseteq B \cap D)$

Assume $(A \subseteq B \land C \subseteq D)$
Assume $x \in A \cap C$
$\therefore x \in A$ and $x \in C$ by the definition of intersection.
$\therefore x \in B$ because $A \subseteq B$.
$\therefore x \in D$ because $C \subseteq D$.
$\therefore x \in B \cap D$ by the definition of intersection.
$\therefore A \cap C \subseteq B \cap D$ by the definition of subset.
$\therefore (A \subseteq B \land C \subseteq D) \Rightarrow (A \cap C \subseteq B \cap D)$ by the definition of implication.

11) Prove theorem T85 on the theorem sheet, using only earlier theorems.

T85 says $\forall_{j\in I}\left(\bigcap_{i\in I} A_i \subseteq A_j\right)$

Let $j \in I$ be arbitrary.
Assume $x \in \bigcap_{i\in I} A_i$
Note that $j \in I$, so the definition of intersection gives us $x \in A_j$.
$\therefore \bigcap_{i\in I} A_i \subseteq A_j$.
$\therefore \forall_{j\in I}\left(\bigcap_{i\in I} A_i \subseteq A_j\right)$.

12) Let $f: A \to B$ and $g: B \to C$ be functions such that $g \circ f: A \twoheadrightarrow C$. Prove that $g: B \twoheadrightarrow C$.
(Hint: Recall that $\twoheadrightarrow$ means surjective)

Let $c \in C$.
There is some $a \in A$ such that $g\big(f(a)\big) = C$.
Note that $f(a) \in B$, which shows that g is surjective.

Part 5: Review (5 points each, 10 points total)

13) Let P and Q be statements. Make the truth table for $P \Rightarrow Q$.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

14) Prove that then $5x > 8$ whenever $x > 2$.

Assume $x > 2$.
$\therefore 5x > 10 > 8$
$\therefore 5x > 8$