Name ____ Test 3, Fall 2021

Part 1: Basic Knowledge (5 points each, 10 points total)

1) Let R be a relation on a set X. Define, precisely, what it means for R to be <u>transitive</u>.

2) In $\mathbb{Z}_{10}$, we would like to relate "5" and "15". Answer the following as using correct (C) notation or incorrect (I) notation.

C I a) $5 \equiv_{10} 15$

C I b) $15 =_{10} 5$

C I c) $5 \equiv 15 \mod 10$

C I d) $15 \equiv 5 \mod 10$

C I e) 5 mod 10 ≡ 15

C I f) 15 mod 10 ≡ 5

C I g) $[5]_{10} = [15]_{10}$

C I h) $(5)_{10} = (15)_{10}$

C I i) $\langle 5 \rangle_{10} = \langle 15 \rangle_{10}$

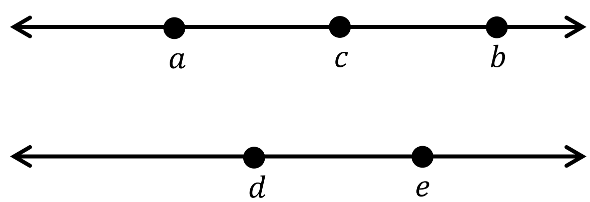
C I j) 15 → 5 conj. 10

Part 2: Basic Skills and Concepts (5 points each, 20 points total)

3) Let $X = \{a, b, c, d\}$ and the relation R be defined by aRa, bRb, cRc, dRd, aRb, and bRa. It is known that R is an equivalence relation, and thus creates a partition on X. Illustrate that partition.

4) Let $X = \{a, b, c, d, e\}$ and construct a partial order relation according to the series of number lines below. What is R? Define it by putting a check (✓) in each entry of the table below where row element relates to the column element. aRb is checked for you as an example.

	a	b	c	d	e
a		✓			
b					
c					
d					
e					



5) Compute the following:

a) $4 \cdot 8 + 5$ in $\mathbb{Z}_7$

b) $6 \cdot 6$ in $\mathbb{Z}_4$

6) Solve for x:

$$3x + 2 \equiv_5 4$$

Part 4: Proofs (10 points each, 50 points total)

7) Let R be the relation defined on $\mathbb{N}$ given by: xRy iff x and y have the same number of 2's in their prime factorization. For example, $48R80$ because $48 = 2^4 \cdot 3$ and $80 = 2^4 \cdot 5$. Prove that R is transitive.

8) Let $\preccurlyeq$ be the relation defined on $\mathbb{N}$ given by: $x \preccurlyeq y$ iff x has fewer or equal to 2's in its prime factorization than y. For example, $36 \preccurlyeq 80$ because $36 = 2^2 \cdot 3^2$, $80 = 2^4 \cdot 5$, and $2 \leq 4$. Prove that $\preccurlyeq$ is *not* antisymmetric.

9) Consider $\mathbb{Z}$ under the “mod n” relation. We know that this relation partitions $\mathbb{Z}$ into n sets:

$$\mathbb{Z}_n = \{[0], [1], [2], \cdots [n-2], [n-1]\}$$

The definition of a partition $\mathcal{P}$ of a set A includes three things, written below. Prove part (c) for the “mod n” relation.

a) If $X \in \mathcal{P}$, then $X \neq \emptyset$
b) If $X \in \mathcal{P}$ and $Y \in \mathcal{P}$ are different sets, then $X \cap Y = \emptyset$.
c) $A = \bigcup_{X \in \mathcal{P}} X$

10) Prove that for all $n \in \mathbb{N}$,

$$\sum_{j=1}^{n} 3^j = \frac{3^{n+1} - 3}{2}$$

11) Prove that for all $n \in \mathbb{N}$,

$$3 + 11 + 19 + \cdots + (8n - 5) = 4n^2 - n$$

Part 5: Review (5 points each, 20 points total)

12) Assume P is true, Q is true, and R is false. What is $(P \vee Q) \Rightarrow (P \wedge R)$?

13) Find the negation of $\forall_{x \in \mathbb{R}} \exists_{y \in \mathbb{Z}} (3x + 2y = 7)$

14) Answer true or false to each of these. Assume $f: \mathbb{R} \to \mathbb{R}$.

T F a) $f(x) = 3x + 2$ is one-to-one.

T F b) $f(x) = 3x + 2$ is onto.

T F c) $f(x) = x^3$ is one-to-one.

T F d) $f(x) = x^3$ is onto.

T F e) $f(x) = \tan^{-1}(x)$ is one-to-one.

15) Let $X = \{1,2,3,4,5\}$ and $Y = (2,4]$. Find $X \cap Y$.

Part 6: Bonus Question (10 bonus points)

16) Prove that for all $n \in \mathbb{N}$,

$$n! \geq 2^{n-1}$$