Name ____ Test 3, Fall 2021

Part 1: Basic Knowledge (5 points each, 10 points total)

1) Let R be a relation on a set X. Define, precisely, what it means for R to be <u>transitive</u>.

For $x, y, z \in X$, if xRy and yRz, then xRz.

2) In $\mathbb{Z}_{10}$, we would like to relate "5" and "15". Answer the following as using correct (C) notation or incorrect (I) notation.

C I a) $5 \equiv_{10} 15$

C I b) $15 =_{10} 5$

C I c) $5 \equiv 15 \mod 10$

C I d) $15 \equiv 5 \mod 10$

C I e) $5 \bmod 10 \equiv 15$

C I f) $15 \bmod 10 \equiv 5$

C I g) $[5]_{10} = [15]_{10}$

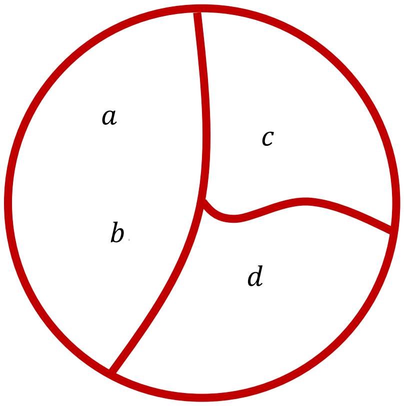
C I h) $(5)_{10} = (15)_{10}$

C I i) $\langle 5 \rangle_{10} = \langle 15 \rangle_{10}$

C I j) $15 \rightarrow 5$ conj. 10

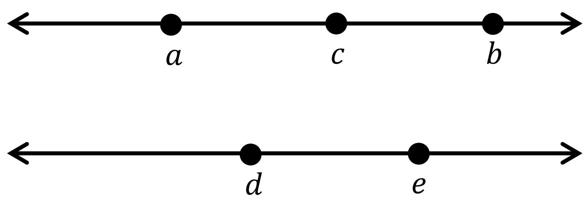
Part 2: Basic Skills and Concepts (5 points each, 20 points total)

3) Let $X = \{a, b, c, d\}$ and the relation R be defined by aRa, bRb, cRc, dRd, aRb, and bRa. It is known that R is an equivalence relation, and thus creates a partition on X. Illustrate that partition.



4) Let $X = \{a, b, c, d, e\}$ and construct a partial order relation according to the series of number lines below. What is R? Define it by putting a check (✓) in each entry of the table below where row element relates to the column element. aRb is checked for you as an example.

	a	b	c	d	e
a	✓	✓	✓		
b		✓			
c		✓	✓		
d				✓	✓
e					✓



5) Compute the following:

a) $4 \cdot 8 + 5$ in $\mathbb{Z}_7$

$4 \cdot 8 + 5 \equiv 4 \cdot 1 + 5 \equiv 4 + 5 \equiv 9 \equiv 2$

b) $6 \cdot 6$ in $\mathbb{Z}_4$

$6 \cdot 6 \equiv 2 \cdot 2 \equiv 4 \equiv 0$

6) Solve for x:

$$3x + 2 \equiv_5 4$$

$3x + 2 \equiv 4$
$3x \equiv 2$

Two methods … brute force it, or realize that $2 \equiv 12$.

Clever method:
$3x \equiv 12$
$x \equiv 4$

(And we know this is the only solution because $\gcd(3,5) = 1$)

Brute force method:

x	$3x$
0	0
1	3
2	$6 \equiv 1$
3	$9 \equiv 4$
4	$12 \equiv 2$

Either way, we see that $x \equiv 4$ is the solution.

Grading note: In modular arithmetic we do not use fractions. Automatic 0 if there's a fraction anywhere in your solution.

Part 4: Proofs (10 points each, 50 points total)

7) Let R be the relation defined on $\mathbb{N}$ given by: xRy iff x and y have the same number of 2's in their prime factorization. For example, $48R80$ because $48 = 2^4 \cdot 3$ and $80 = 2^4 \cdot 5$. Prove that R is transitive.

Let $x, y, z \in \mathbb{N}$ and assume xRy and yRz.

Because xRy:
$x = 2^a b$, $y = 2^a c$ for some $a, b, c \in \mathbb{N}$ with $\gcd(2, b) = \gcd(2, c) = 1$.

Because yRz:
$y = 2^d e$, $z = 2^d f$ for some $d, e, f \in \mathbb{N}$ with $\gcd(2, e) = \gcd(2, f) = 1$.

Note that $y = 2^a c = 2^d e$, so $a = d$ and $c = e$.

Thus:
$x = 2^a b$ and $z = 2^a f$.

Therefore xRz, so R is transitive.

You could also do this as an informal proof with more words and fewer equations:

Assume xRy and yRz. Note that xRy means that x and y have the same number of 2's in their prime factorization. Similarly y and z have the same number of 2's in their prime factorization. Because y appears in both of these, all three have the same number of 2's in their prime factorization. In particular x and z have the same number of 2's in their prime factorization, which is xRz. Therefore R is transitive.

8) Let $\preccurlyeq$ be the relation defined on $\mathbb{N}$ given by: $x \preccurlyeq y$ iff x has fewer or equal to 2's in its prime factorization than y. For example, $36 \preccurlyeq 80$ because $36 = 2^2 \cdot 3^2$, $80 = 2^4 \cdot 5$, and $2 \leq 4$. Prove that $\preccurlyeq$ is *not* antisymmetric.

Consider $x = 8$ and $y = 24$.
$x = 2^3 \cdot 1$
$y = 2^3 \cdot 3$

Note that $x \preccurlyeq y$ because $3 \leq 3$. Similarly $y \preccurlyeq x$. However, $x \neq y$, so $\preccurlyeq$ is not antisymmetric.

Note: I used the symbol $\preccurlyeq$ thinking this was a partial order relation … it's not. As is obvious from the fact that you proved it's not antisymmetric. Doesn't affect the problem at all, but you should know that notation like this should only be used for partial order relations.

9) Consider $\mathbb{Z}$ under the "mod n" relation. We know that this relation partitions $\mathbb{Z}$ into n sets:

$$\mathbb{Z}_n = \{[0], [1], [2], \cdots [n-2], [n-1]\}$$

The definition of a partition $\mathcal{P}$ of a set A includes three things, written below. Prove part (c) for the "mod n" relation.

a) If $X \in \mathcal{P}$, then $X \neq \emptyset$
b) If $X \in \mathcal{P}$ and $Y \in \mathcal{P}$ are different sets, then $X \cap Y = \emptyset$.
c) $A = \bigcup_{X \in \mathcal{P}} X$

In language of our relation, this says:

$$\mathbb{Z} = \bigcup_{[x] \in \mathbb{Z}_n} [x]$$

Obviously, $\bigcup_{[x] \in \mathbb{Z}_n} [x] \subseteq \mathbb{Z}$.

On the other hand, let $a \in \mathbb{Z}$.

$\therefore a \in [a]$

$\therefore a \in \bigcup\limits_{[x] \in \mathbb{Z}_n} [x]$

$\therefore \mathbb{Z} \subseteq \bigcup\limits_{[x] \in \mathbb{Z}_n} [x]$

$\therefore \bigcup\limits_{[x] \in \mathbb{Z}_n} [x] = \mathbb{Z}$

10) Prove that for all $n \in \mathbb{N}$,

$$\sum_{j=1}^{n} 3^j = \frac{3^{n+1} - 3}{2}$$

Base case: $n = 1$:

$$\sum_{j=1}^{1} 3^j = 3^1 = 3$$

$$\frac{3^{1+1} - 3}{2} = \frac{9 - 3}{2} = \frac{6}{2} = 3$$

$$\therefore \sum_{j=1}^{1} 3^j = \frac{3^{1+1} - 3}{2}$$

Induction Hypothesis: Let $k \in \mathbb{N}$ and assume:

$$\sum_{j=1}^{k} 3^j = \frac{3^{k+1} - 3}{2}$$

$$\sum_{j=1}^{k+1} 3^j = \sum_{j=1}^{k} 3^j + 3^{k+1} = \frac{3^{k+1} - 3}{2} + 3^{k+1} = \frac{3^{k+1} - 3}{2} + \frac{2 \cdot 3^{k+1}}{2} = \frac{3^{k+1} - 3 + 2 \cdot 3^{k+1}}{2}$$
$$= \frac{3 \cdot 3^{k+1} - 3}{2} = \frac{3^{k+2} - 3}{2}$$

Therefore for all $n \in \mathbb{N}$:

$$\sum_{j=1}^{n} 3^j = \frac{3^{n+1} - 3}{2}$$

Grading note:

The base case and induction hypothesis were considered one 5pt problem.

The induction step was considered another 5pt problem.

The final conclusion was marked but not given a grade.

11) Prove that for all $n \in \mathbb{N}$,

$$3 + 11 + 19 + \cdots + (8n - 5) = 4n^2 - n$$

First let's rewrite this in terms of summations, because I find it so much easier to deal with:

$$\sum_{j=1}^{n} 8j - 5 = 4n^2 - n$$

Base case:

$$\sum_{j=1}^{1} 8j - 5 = 8 - 5 = 3 = 4 - 1 = 4 \cdot 1^2 - 1$$

Induction Hypothesis: For some $k \in \mathbb{N}$, assume:

$$\sum_{j=1}^{k} 8j - 5 = 4k^2 - k$$

$$\therefore \sum_{j=1}^{k+1} 8j - 5 = \left(\sum_{j=1}^{k} 8j - 5 \right) + (8(k+1) - 5) = 4k^2 - k + (8k + 8 - 5) = 4k^2 - k + 8k + 3$$

$$= 4k^2 + 7k + 3 = 4k^2 + 8k + 4 - (k+1) = 4(k^2 + 2k + 1) - (k+1) = 4(k+1)^2 - (k+1)$$

Therefore, by induction, for all $n \in \mathbb{N}$, assume:

$$\sum_{j=1}^{n} 8j - 5 = 4n^2 - n$$

Grading note:

The base case and induction hypothesis were considered one 5pt problem.

The induction step was considered another 5pt problem.

The final conclusion was marked but not given a grade.

Part 5: Review (5 points each, 20 points total)

12) Assume P is true, Q is true, and R is false. What is $(P \vee Q) \Rightarrow (P \wedge R)$?

$P \vee Q$ is true, but $P \wedge R$ is false, so $(P \vee Q) \Rightarrow (P \wedge R)$ is false.

13) Find the negation of $\forall_{x\in\mathbb{R}}\exists_{y\in\mathbb{Z}}(3x + 2y = 7)$

$$\sim\forall_{x\in\mathbb{R}}\exists_{y\in\mathbb{Z}}(3x + 2y = 7) = \exists_{x\in\mathbb{R}}\forall_{y\in\mathbb{Z}}(3x + 2y \neq 7)$$

14) Answer true or false to each of these. Assume $f: \mathbb{R} \to \mathbb{R}$.

T F a) $f(x) = 3x + 2$ is one-to-one.

T F b) $f(x) = 3x + 2$ is onto.

T F c) $f(x) = x^3$ is one-to-one.

T F d) $f(x) = x^3$ is onto.

T F e) $f(x) = \tan^{-1}(x)$ is one-to-one.

15) Let $X = \{1,2,3,4,5\}$ and $Y = (2,4]$. Find $X \cap Y$.

$\{3,4\}$

Part 6: Bonus Question (10 bonus points)

16) Prove that for all $n \in \mathbb{N}$,

$$n! \geq 2^{n-1}$$

Base case (n=1):

$1! = 1 \geq 1^1 = 2^0 = 2^{1-1}$

Assume $k! \geq 2^{k-1}$ for some $k \in \mathbb{N}$

$$(k+1)! = (k+1)k! \geq (k+1)2^{k-1} \geq (1+1)2^{k-1} = 2 \cdot 2^{k-1} = 2^k$$

Therefore by induction, for all $n \in \mathbb{N}$:

$$n! \geq 2^{n-1}$$