Name ____ Test 1, Fall 2022

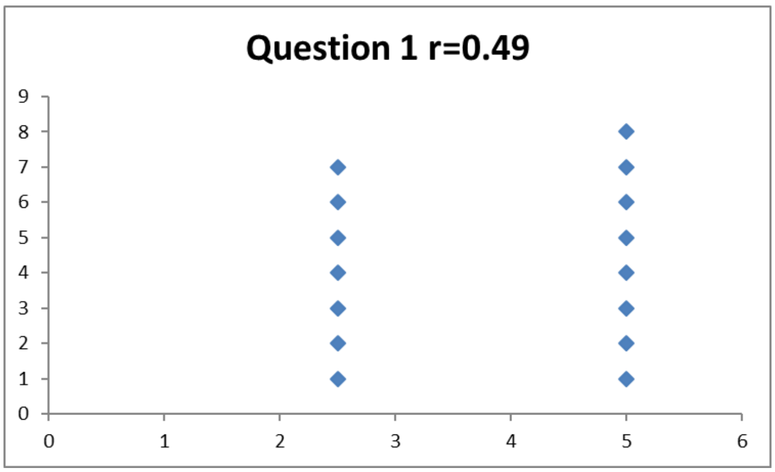
Throughout the test, unless otherwise specified, you may assume P, Q, and R are statements, x, y, and z are real numbers, m, n, r and s are integers.

Part 1: Basic Knowledge

1) Give a precise definition of an <u>odd</u> integer. (5 points)

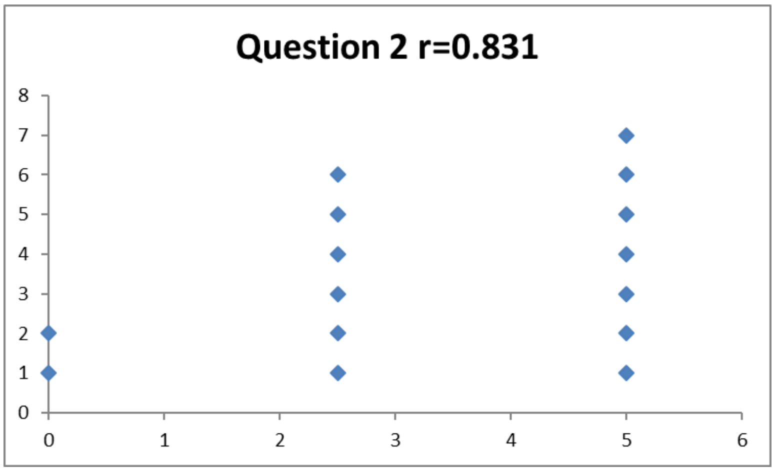
n is odd if $n = 2k + 1$ for some integer k.

n is odd if $\exists_{k\in\mathbb{Z}}(n = 2k + 1)$



2) Give a precise definition of an <u>open statement</u>. (5 points)

A sentence or mathematical expression with a variable. When a value is substituted for the variable, it becomes a statement.



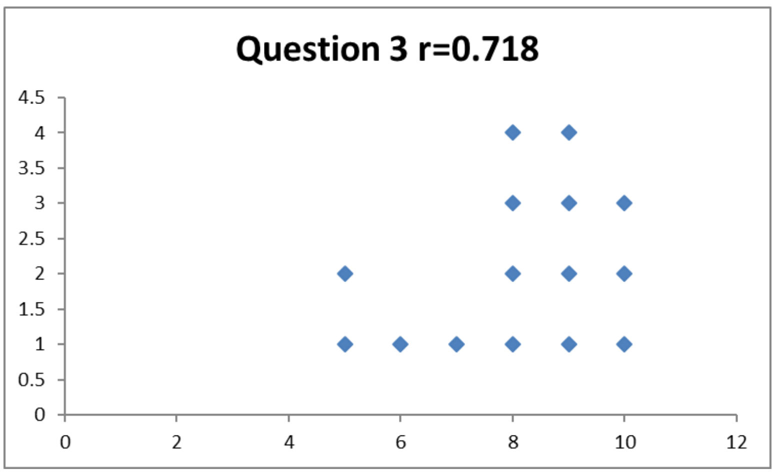
3) Answer each of the following. (1 points each)

(T) F a) This is a universal statement: Every cat has nine lives.
T (F) b) This is a universal statement: Fluffly has nine lives.
T (F) c) This is a universal statement: Whenever fluffy falls, she expends a life.

T (F) d) This is an existential statement: If you score an A, you have done well.
(T) F e) This is an existential statement: There is somebody that has scored an A.
T (F) f) This is an existential statement: Michael scored an A.

T (F) g) $P \Rightarrow Q$ is equivalent to $Q \Rightarrow P$
(T) F h) $P \Rightarrow Q$ is equivalent to $\sim Q \Rightarrow \sim P$
T (F) i) $P \Rightarrow Q$ is equivalent to $\sim P \Rightarrow \sim Q$

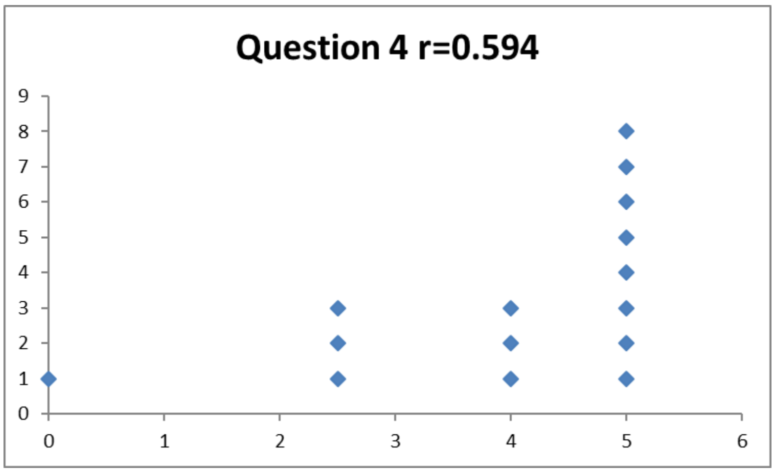
(T) F j) $P \vee \sim P$



Part 2: Basic Skills and Concepts (5 points each, 20 points total)

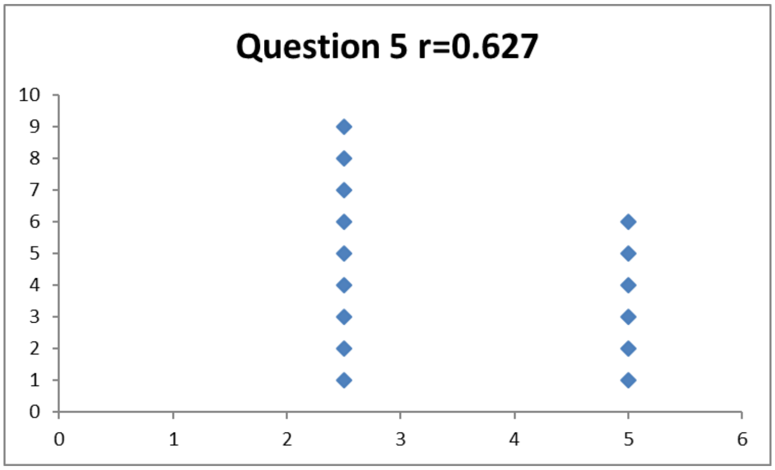
4) Give the complete truth table for $P \Rightarrow \sim(Q \vee P)$.

P	Q	$Q \vee P$	$\sim(Q \vee P)$	$P \Rightarrow \sim(Q \vee P)$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	T
F	F	F	T	T



5) Simplify $\sim(P \wedge (P \Rightarrow Q))$ so that negations are applied only to individual statements.

$$\sim(P \wedge (P \Rightarrow Q))$$
$$\sim P \vee \sim(P \Rightarrow Q)$$
$$\sim P \vee (P \wedge \sim Q)$$

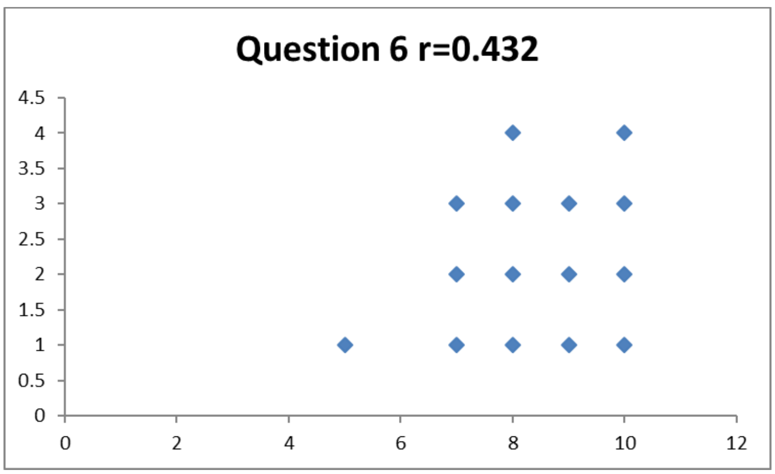


6) Answer each of the following. A true/false question is true if it is *always* true and false if it is *ever* false. (1 points each)

(T) F a) If $x > 0$, then $|x| = x$
(T) F b) If $x < 0$, then $|x| = -x$
(T) F c) $|x| \geq 0$

(T) F d) If I know a statement is true, it must be true.
T (F) e) If a statement is true, some person must know it is true.
T (F) f) Mathematics proves a "God" exists because there must be must be some being that knows the truth value of all statements. We call that being "God".

T (F) g) $\forall_{x\in\mathbb{R}}\forall_{y\in R}(x+y=5)$
(T) F h) $\forall_{x\in\mathbb{R}}\exists_{y\in R}(x+y=5)$
T (F) i) $\exists_{x\in\mathbb{R}}\forall_{y\in R}(x+y=5)$
(T) F j) $\exists_{x\in\mathbb{R}}\exists_{y\in R}(x+y=5)$



Part 3: Proofs (10 points each, 60 points total)

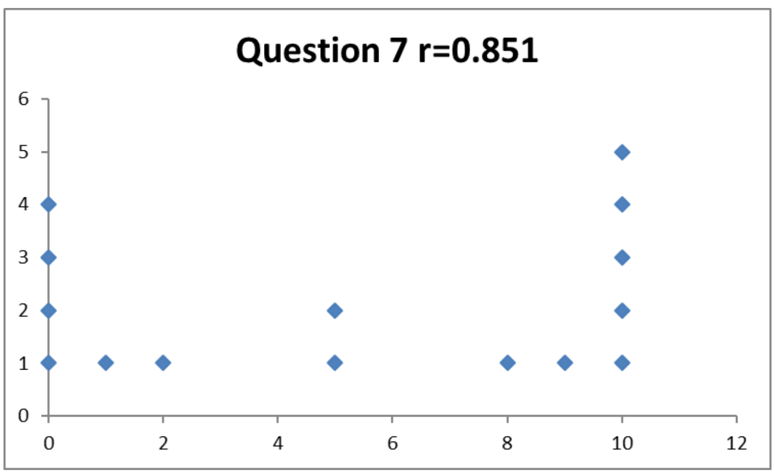
7) Prove that for all integers n, 1 divides n. That is,

$$\forall_{n \in \mathbb{Z}}(1|n)$$

Let m be an integer.	
$\therefore m = m \cdot 1$	Algebra
$\therefore 1|m$	Definition of divides.
$\therefore 1|n$ for all integers n.	Universal Generalization

OR

Let n be an integer. Then $m = m \cdot 1$, which satisfies the definition of divides. Therefore $1|m$. Because m was arbitrary, we may generalize it to say that $1|n$ for all integers n.



8) Let x and y be integers. Prove that if x is odd and y is even, then $3x + 4y$ is odd.

Assume x is odd and y is even.

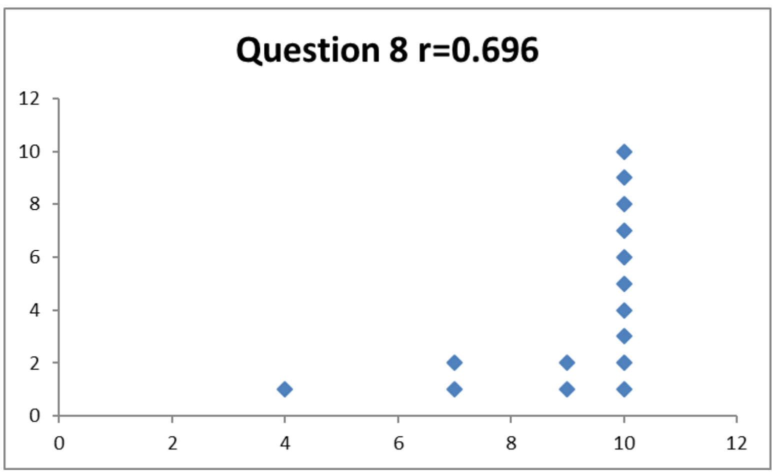
$\therefore x = 2k + 1$ for some $k \in \mathbb{Z}$	Definition of odd.
$\therefore y = 2k_2$ for some $k_2 \in \mathbb{Z}$	Definition of even.
$\therefore 3x + 4y = 3(2k + 1) + 4(2k_2)$	
$= 6k + 3 + 8k_2$	Plug in & simplify
$= 2(3k + 4k_2 + 1) + 1$	
$\therefore 3x + 4y$ is odd.	Definition of odd.

OR

Assume x is odd and y is even. Then $x = 2k + 1$ and $y = 2k_2$ for some integers k, k_2. Plug these into our desired expression to get:

$$3x + 4y = 3(2k + 1) + 4(2k_2) = 6k + 3 + 8k_2 = 2(3k + 4k_2 + 1) + 1$$

Therefore by the definition of odd, we see that $3x + 4y$ is odd.



9) Let x be a real number. Prove that if x is rational, then twice x is also rational. That is,

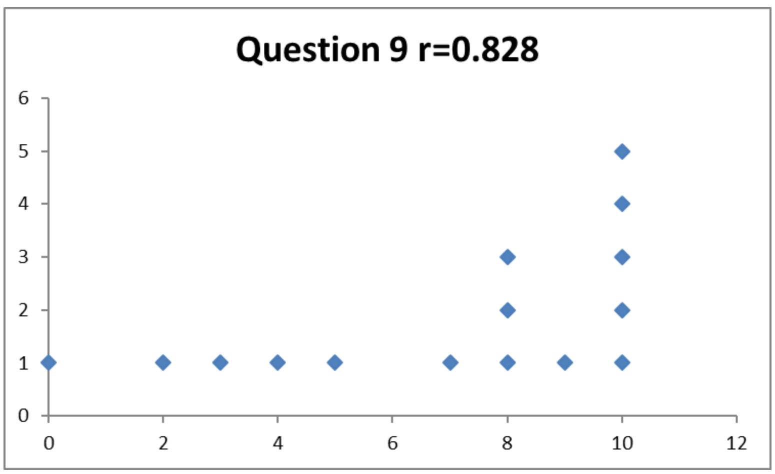
$$x \in \mathbb{Q} \Rightarrow 2x \in \mathbb{Q}$$

Assume x is rational. Then we may write $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$. Therefore $2x = \frac{2a}{b}$ which also satisfies the definition of rational because $2a$ is also an integer.

OR

Assume $x \in \mathbb{Q}$

$\therefore x = \frac{a}{b}$ for some $a, b\ in\ \mathbb{Z}$	Definition of rational.
$\therefore 2x = \frac{2a}{b}$	Multiply the above by 2.
$\therefore 2x \in \mathbb{Q}$	Definition of rational.



10) Let x be a real number. If $x^3 + 2x^2 < 0$, then $2x + 5 < 11$

Assume $x^3 + 2x^2 < 0$.

$\therefore x^2(x+2) < 0$	Factor
$\therefore x + 2 < 0$	Positivity axiom and the fact that $x^2 > 0$
$\therefore 2x + 4 < 0$	Multiply the above by 2.
$\therefore 2x + 5 < 1$	Add 1 to each sides of the above.
$\therefore 2x + 5 < 11$	Transitive property on the above and $1 < 11$.

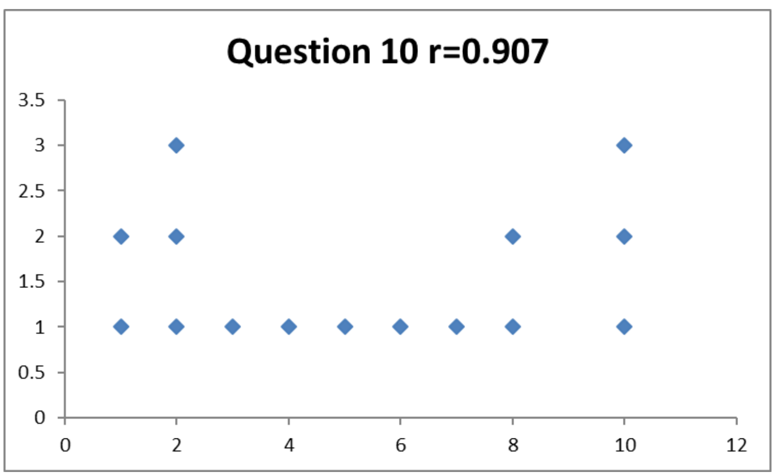
OR

Assume $x^3 + 2x^2 < 0$. Note that $x^3 + 2x^2 = x^2(x+2)$ and that $x^2 > 0$. Therefore by the positivity axiom we get $x + 2 < 0$. Some basic algebra then gives us:

$$2x + 5 = (x+2) + (x+2) + 1 < 0 + 0 + 1 < 11$$

OR use the contrapositive – I think most high scoring proofs did it that method, they look something like this:

$2x + 5 \geq 11$
[math]
$\therefore x \geq 3$
$\therefore x^3 + 2x^2 \geq [\text{math}] \geq 0$



11) Consider this supposed theorem: “If n is even, then lightsabers are real”. We may have a difficult time proving this theorem, but it is hopefully pretty clear what the first two lines and last line of such a proof would likely be. Fill in those blanks below. You need not provide justification.

Proof that if n is even, then lightsabers are real

1) Assume n is even.

2) $\therefore n = 2k$ for some $k \in \mathbb{Z}$

⋮

[Intermediate steps are omitted]

⋮

17) $\therefore$ lightsabers are real.

Now suppose we have a difficult time proving this theorem, so we turn to the *contrapositive*! We may still have a difficult time proving the theorem, but again it should be clear what the first and last two lines of such a proof would likely be. Fill in those blanks below. You need not provide justification.

Proof that if n is even, then lightsabers are real

1) Lightsabers at not real.

⋮

[Intermediate steps are omitted]

⋮

13) $\therefore n$ is not even.

14) Therefore, if n is even, then lightsabers are real

2-point Bonus: What should would 12 in the above proof likely be?

12) n is odd or $n = 2(\cdots) + 1$