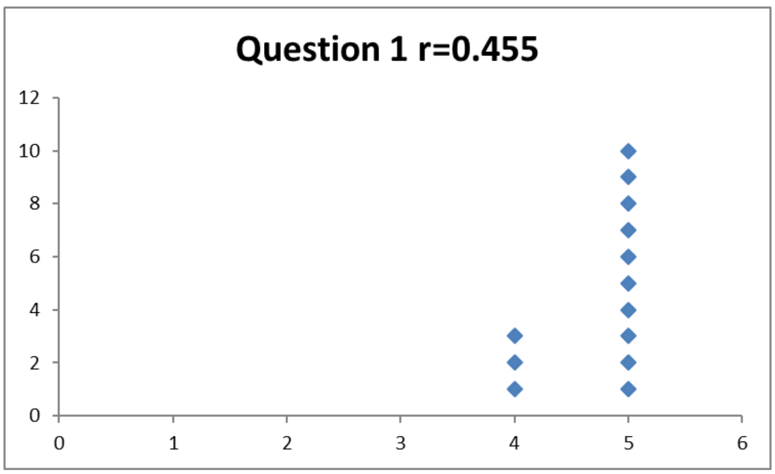
Name ____ Test 2, Fall 2022

Throughout the test, unless otherwise specified, you may assume P, Q, and R are statements, x, y, and z are real numbers, m, n, r and s are integers.

Part 1: Basic Knowledge

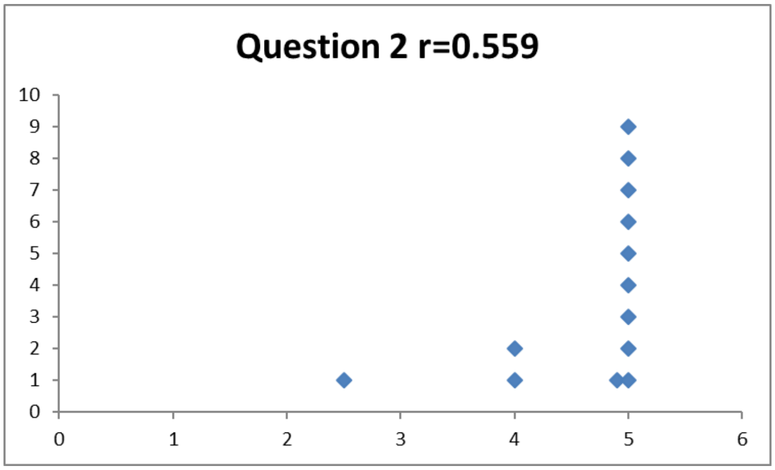
1) Give a precise definition of an <u>intersection</u> of sets A and B. (5 points)

$$A \cap B := \{x \in U | x \in A \text{ and } x \in B\}$$



2) Give a precise definition of the complement of a set A. (5 points)

$$A^c := \{x \in U | x \notin A\}$$



Part 2: Basic Skills and Concepts

3) Answer each of the following. (1 points each)

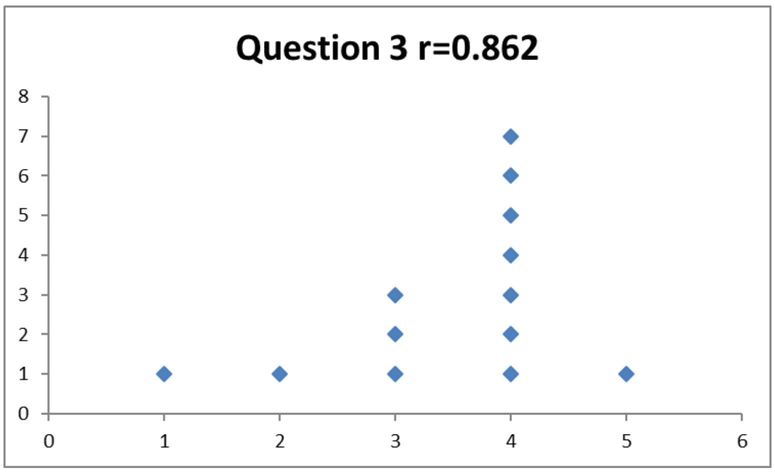
T F a) Coding in LaTeX requires working with three files: Source (.tex), Goop (.goo), and Output (.pdf).

T F b) LaTeX is "What You See is What You Get", just like Microsoft Word.

T F c) LaTeX applies order of operations, just like a calculator.

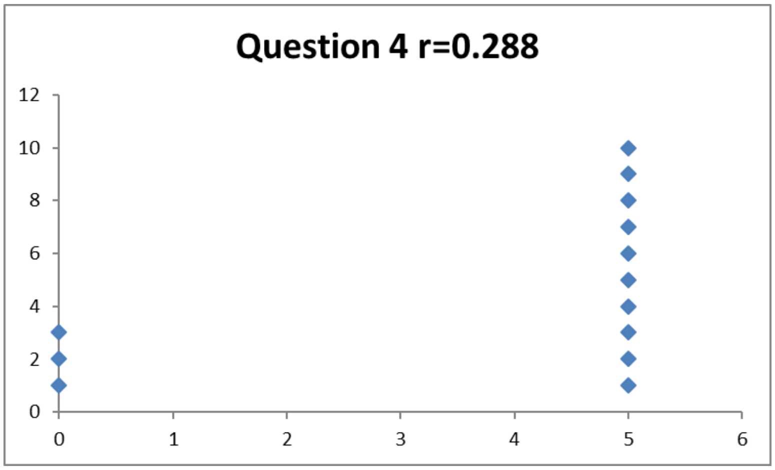
T F d) Every math major will use LaTeX in their career after school.

T F e) In LaTeX, dollar signs ($) are used to create inline math, while ampersands (&) are used to create display math.



4) What types of objects can be elements of sets? (5 points)

(A) Numbers
(B) Letters
(C) Sets
(D) Both (A) and (B)
(E) All of the above



5) Let $A = [0,5]$ and $B = (2,10)$ in the universe $\mathbb{R}$. (5 points)

(A) What is $A \cup B$?

$$[0,10)$$

(B) What is $A \cap B$?

$$(2,5]$$

(C) What is $A - B$?

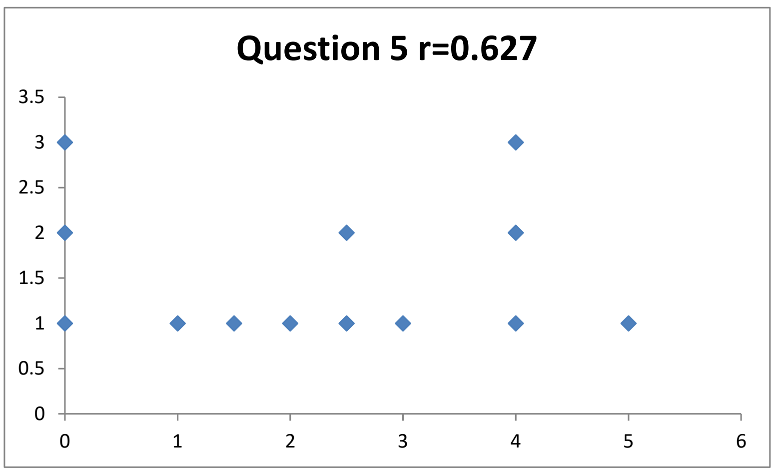
$$[0,2]$$

(D) What is A^c?

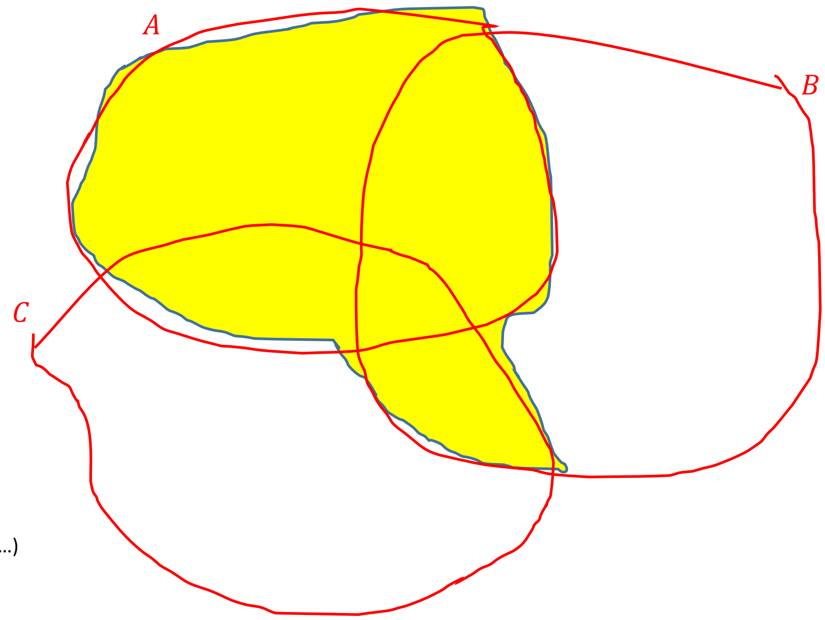
$$(-\infty, 0) \cup (5, \infty)$$

(E) How many elements does $\mathcal{P}(A)$ contain?

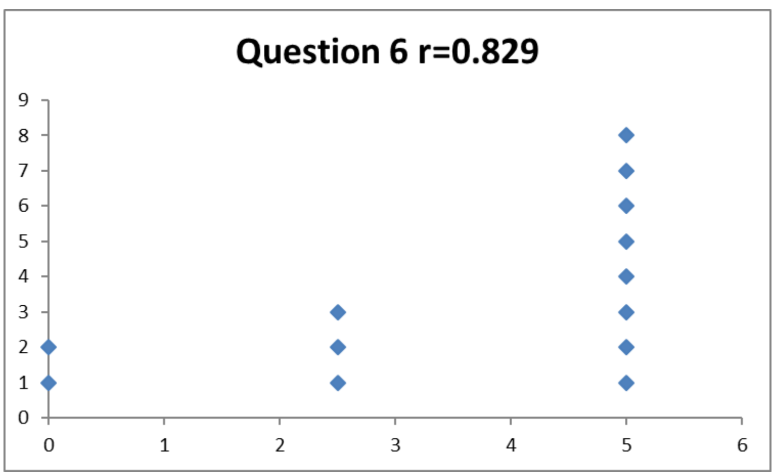
$$\infty$$



6) Draw an illustration of $A \cup (B \cap C)$. (5 points)



(Dr. Beyerl is not an artist…)



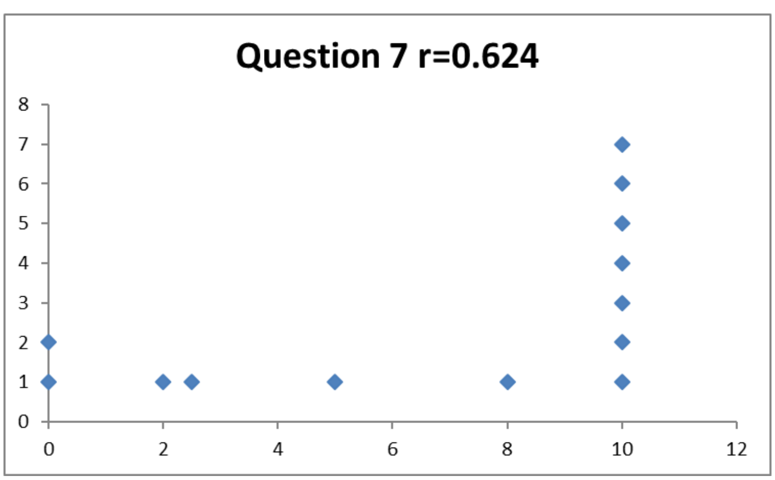
Part 3: Proofs (10 points each, 60 points total)

7) Below is a proof that $\sqrt{6} \notin \mathbb{Q}$. Well, maybe. I just copied and pasted the proof we did that $\sqrt{5} \notin \mathbb{Q}$ and replaced 5 with 6. You should be wary of anyone that claims this proof is still valid. Maybe it is, maybe it isn't. But one thing is for sure, one step(s) requires better justification than we've seen with $\sqrt{5}$ because the same reasoning doesn't work.

Which is the step(s) that *should* not be clear to a student in this class, such as you? Circle that step and explain why it's not clear.
(You don't need to justify it. We haven't covered that. I want to know why the reasoning from the other proof doesn't work.)

1) Assume $\sqrt{6} \in \mathbb{Q}$
2) $\sqrt{6} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$
3) WLOG $\gcd(a, b) = 1$
4) $\sqrt{6} \cdot b = a$
5) $6b^2 = a^2$
6) $6|a^2$
7) $6|a$
8) $a = 6k$ for some $k \in \mathbb{Z}$
9) $6b^2 = (6k)^2$
10) $6b^2 = 36k^2$
11) $b^2 = 6k^2$
12) $6|b^2$
13) $6|b$
14) $\gcd(a, b) \geq 6$
15) $\sqrt{6} \notin \mathbb{Q}$

Why does $6|a$? In the other proof we used the fact that 5 was prime to get $5|a$. However, 6 is not prime.



8) Prove theorem T52 on the theorem sheet, without using T52 itself or any later theorems.

Theorem: $(A^c)^c = A$

($\subseteq$) Assume $x \in (A^c)^c$. By definition of complement, $x \notin A^c$. Then again by the same definition, $x \in A$. Thus $(A^c)^c \subseteq A$.

($\supseteq$) Assume $x \in A$. By definition of complement, $x \notin A^c$. This means precisely that $x \in (A^c)^c$, so we get $A \subseteq (A^c)^c$.

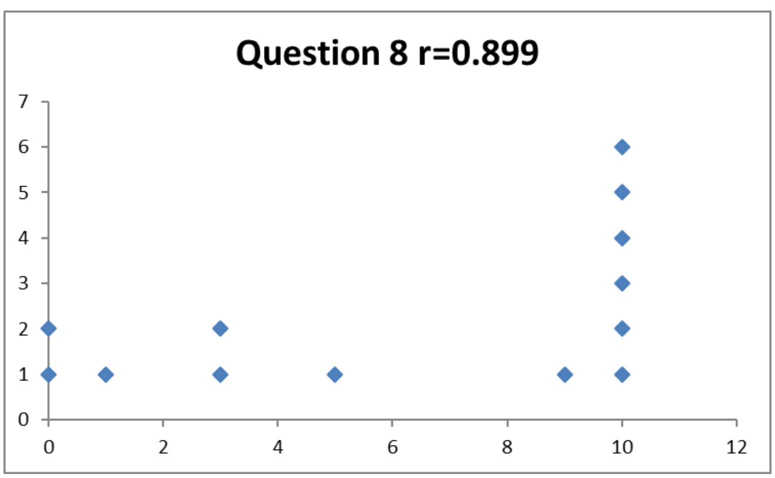
Therefore $(A^c)^c = A$.

OR as a two-column proof, here are the steps, you would also need to provide appropriate justification for the steps that are not assumptions.

Assume $x \in (A^c)^c$
 $x \notin A^c$
 $x \in A$
$(A^c)^c \subseteq A$

Assume $x \in A$
 $x \notin A^c$
 $x \in (A^c)^c$
$A \subseteq (A^c)^c$

$(A^c)^c = A$



9) Prove theorem T67 on the theorem sheet, without using T67 itself or any later theorems.

Theorem: $(A \subseteq B \wedge C \subseteq D) \Rightarrow (A \cup C) \subseteq (B \cup D)$

Assume $A \subseteq B$ and $C \subseteq D$. Further suppose that $x \in A \cup C$, which means that either $x \in A$ or $x \in C$. This creates two cases.

Case 1) $x \in A$. In this case use $A \subseteq B$ to get $x \in B$. Thus $x \in B \cup D$.

Case 2) $x \in C$. In this case use $C \subseteq D$ to get $x \in D$. Thus again $x \in B \cup D$.

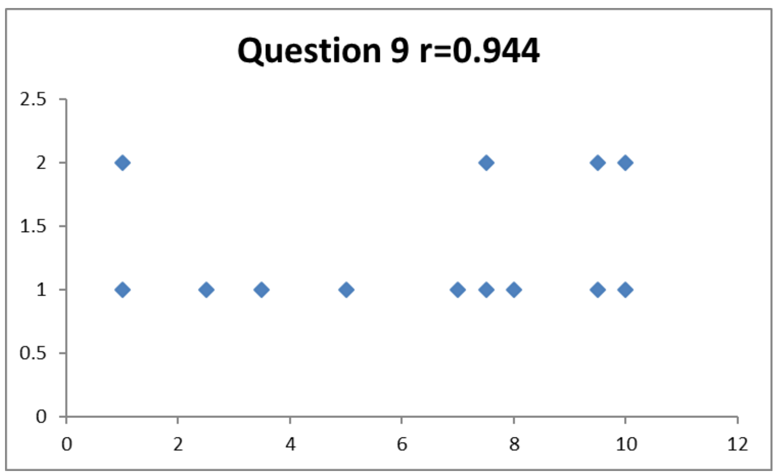
In either case $x \in B \cup D$, therefore $(A \cup C) \subseteq (B \cup D)$.

OR as a two-column proof, here are the steps, you would also need to provide appropriate justification for the steps that are not assumptions.

Assume $A \subseteq B$
Assume $C \subseteq D$
Assume $x \in A \cup C$
Case 1: $x \in A$
 $x \in B$
 $x \in B \cup D$
Case 2: $x \in C$
 $x \in D$
 $x \in B \cup D$
$x \in B \cup D$
$(A \cup C) \subseteq (B \cup D)$

OR

Assume $A \subseteq B$
Assume $C \subseteq D$
Assume $x \in A \cup C$
Case 1: $x \in A$
 $x \in B$
 $x \in B \cup D$
 $(A \cup C) \subseteq (B \cup D)$
Case 2: $x \in C$
 $x \in D$
 $x \in B \cup D$
 $(A \cup C) \subseteq (B \cup D)$
$(A \cup C) \subseteq (B \cup D)$

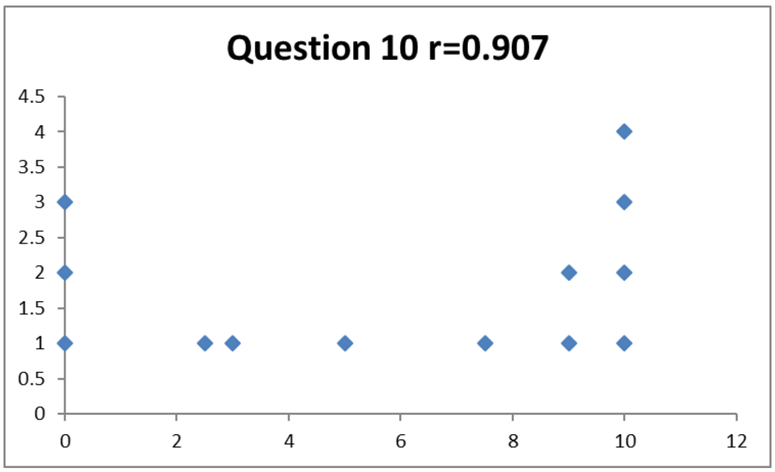


10) Prove that if $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

Assume $A \subseteq B$ and further suppose that $X \in \mathcal{P}(A)$. Thus by the definition of power set $X \subseteq A$. So also we then get $X \subseteq B$ because $A \subseteq B$. Thus $X \in \mathcal{P}(B)$, and so therefore $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

OR as a two-column proof, here are the steps, you would also need to provide appropriate justification for the steps that are not assumptions.

Assume $A \subseteq B$
Assume $X \in \mathcal{P}(A)$
$X \subseteq A$
$X \subseteq B$
$X \in \mathcal{P}(B)$
$\mathcal{P}(A) \subseteq \mathcal{P}(B)$



11) Prove that $x^2 = -4$ has no real solutions.

Assume $x^2 = -4$ has a real solution. Call that solution a, so $a^2 = -4$. Note that $-4 < 0$, so $a^2 < 0$. There are three cases to consider.

Case 1) $a > 0$. By the positivity axiom, $a \cdot a > 0$, which results in a contradiction with $a^2 < 0$.

Case 2) $a < 0$. Here note that $-a > 0$ and $(-1)^2 = 2$, so again by the positivity axiom we get:

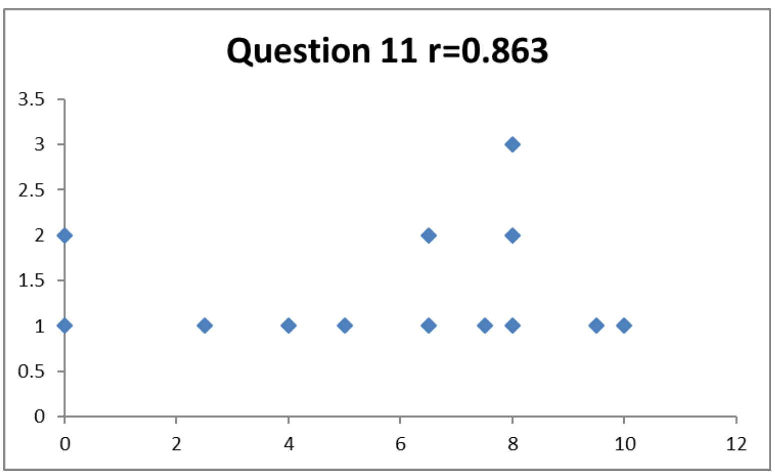
$$a^2 = 1a^2 = (-1)^2 a^2 = (-a)(-a) > 0$$

This is also a contradiction with $a^2 < 0$.

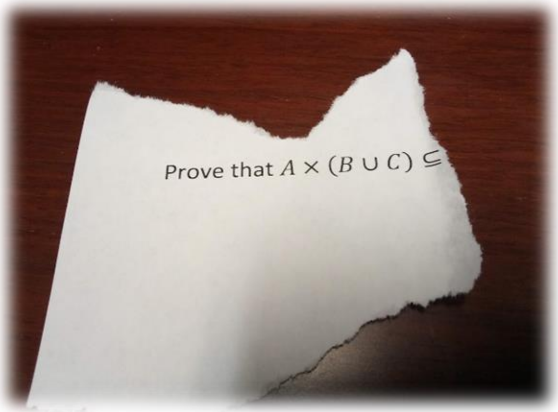
Case 3) $a = 0$. In this case $a^2 = 0$, which is a contradiction with $a^2 = -4$.

In each case we get a contradiction, so there can be no solutions to $x^2 = -4$.

This proof is kind of difficult to prove because it's so obvious. We know that $x^2 > 0$, but how do we justify that? Ultimately it goes back to the positivity axiom that the product of two positive numbers is positive. What I'm looking for is the structure of the poof, generous partial credit is given as long as you set it up as a proof by contradiction, demonstrate a contradiction, and make the conclusion.



12) Alice is taking a set theory course, but unfortunately her dog ate her homework assignment! Pictured here is what is left over. Help Alice out by setting up as much of the structure and logic of this proof as you can, without knowing what the rest of the statement is.

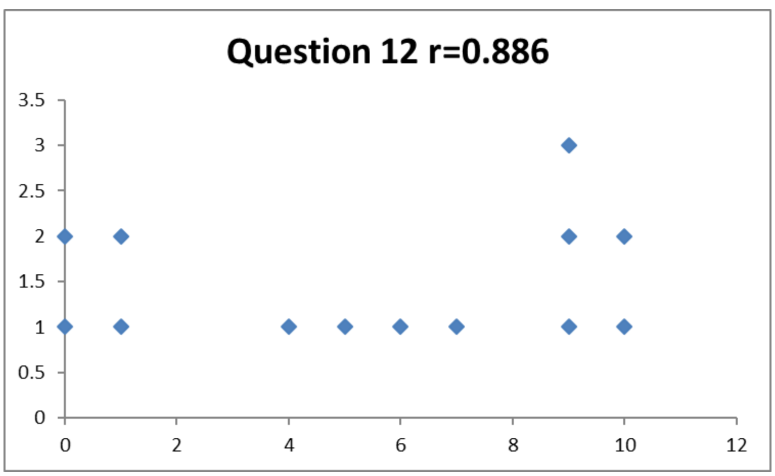


Assume $x \in A \times (B \cup C)$
$x = (y, z)$ where $y \in A$ and $z \in B \cup C$

Case 1) $z \in B$
Math

Case 2) $z \in C$
Math

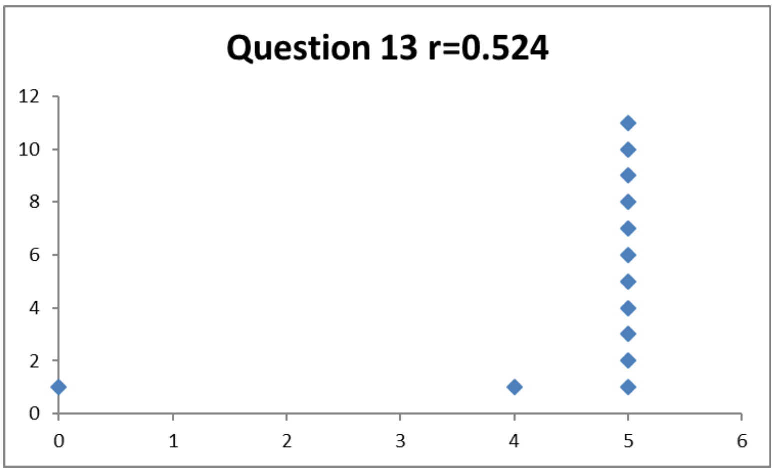
Conclusion



Part 4: Review

13) Give the truth table for $P \Rightarrow Q$. (5 points)

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
T	F	T



14) Find the negation of $\forall_{x\in\mathbb{R}}\exists_{y\in\mathbb{R}}(x+y=5)$. (5 points)

$\exists_{x\in\mathbb{R}}\forall_{y\in\mathbb{R}}(x+y\neq 5)$

OR

"False"
…Because the statement itself is true so the negation of it is false. I doubt anybody will do that though, so I'll leave this question as is and see what happens … you would have to be very confident in yourself and your grade to give an answer that is mathematically correct but so obviously not what the professor is asking for. But correct is correct, it doesn't matter what you think I'm asking for, only what I'm actually asking for.