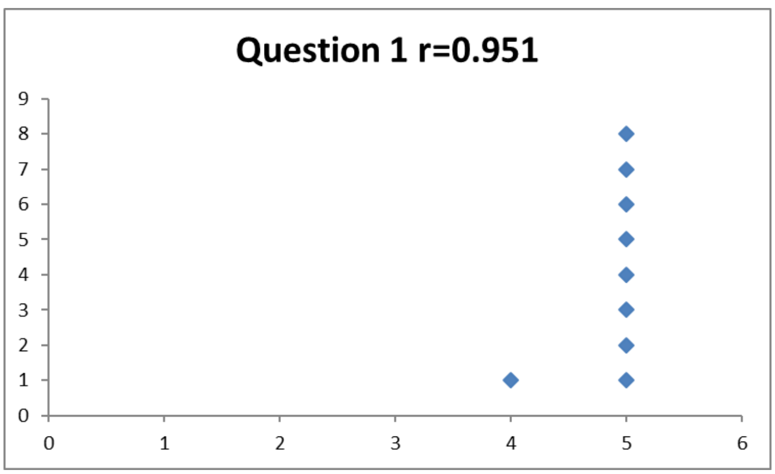
Name ____ Test 3, Fall 2022

Note that 331 is prime, in case that is relevant at some point.

Part 1: Basic Knowledge

1) What does $a \equiv b \bmod m$ mean? Hint: Don't write a sentence, just give the mathematical "tool" (5 points)

$$m|a-b$$



2) Let $f: A \to B$ be a function. What does it mean for f to be surjective? Give a precise definition. (5 points)
Hint: onto is a synonym of surjective.

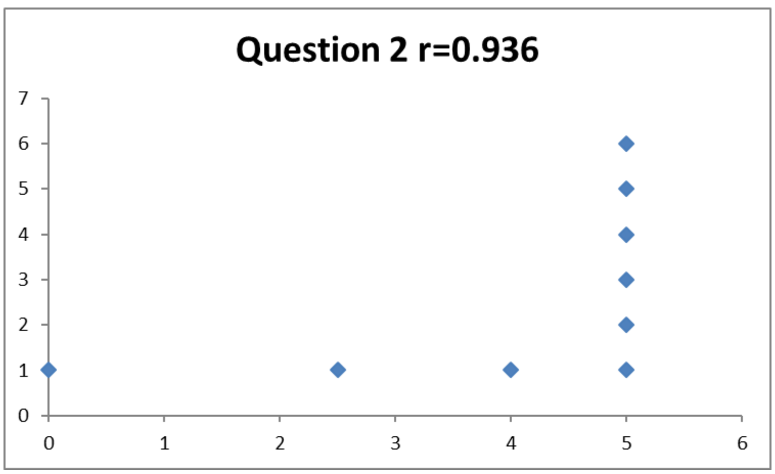
For every $b \in B$, there is an $a \in A$ such that $f(a) = b$

OR

$\forall_{b \in B} \exists_{a \in A} (f(a) = b)$

OR

For every element in the codomain B, there is something in the domain A that maps to it via f.



Part 2: Basic Skills and Concepts

3) Answer each of the following. (1 points each)

T F a) $6x \equiv_{300} 1$ has a exactly 1 solution.

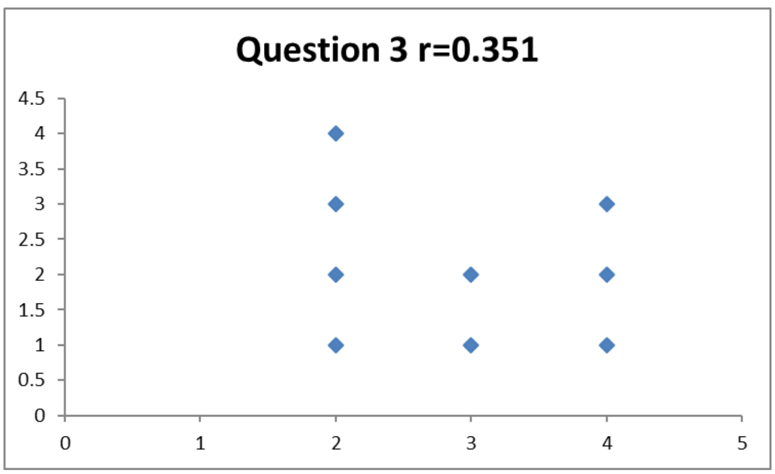
T F b) $6x \equiv_{300} 12$ has exactly 1 solution.

T F c) $6x \equiv_{331} 1$ has exactly 1 solution.

T F d) If $a \equiv_{331} b$, then $a \equiv_{662} b$.

T F e) If $a \equiv_{662} b$, then $a \equiv_{331} b$.

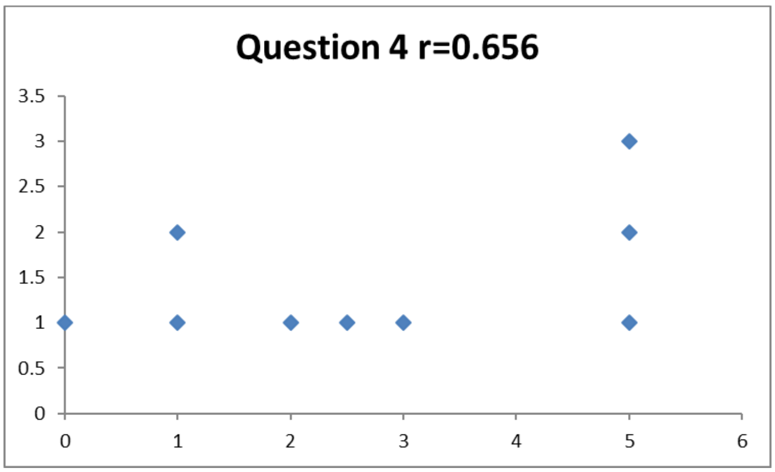
Look at mod 2 and 4 to make sense of these last two.



4) Find the intersection below. (5 points)

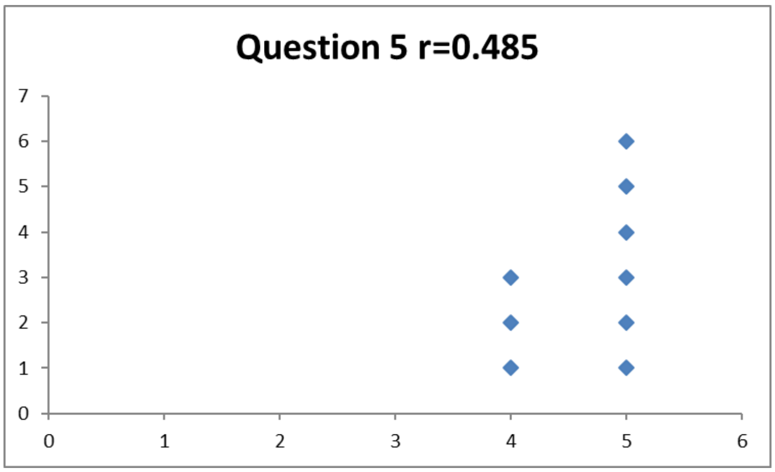
$$\bigcap_{k=1}^{\infty}\left[\frac{1}{k}, 5+\frac{1}{k}\right)$$

$[1,6) \cap [0.5,5.5) \cap [0.\overline{3}, 5.\overline{3}) \cap \cdots = [1,5]$



5) Solve $3x + 2 \equiv 9 \bmod 10$ (5 points)

$3x \equiv 7$
$7 \cdot 3x \equiv 7 \cdot 7$
$x \equiv 49$
$x \equiv 9$



6) Solve $2x + 8 \equiv 4 \text{ mod } 10$ (5 points)

$2x \equiv -4$
$2x \equiv 6$

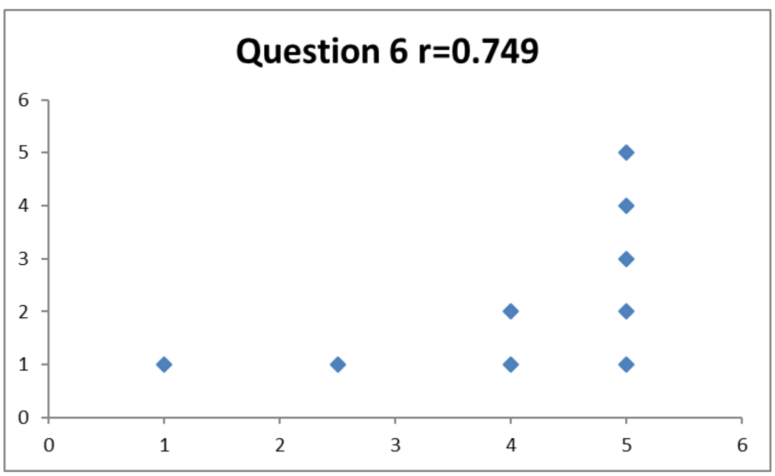
x	$2x$
0	0
1	2
2	4
3	6
4	8
5	0
6	2
7	4
8	6
9	8

$x \equiv 3, 8$

OR

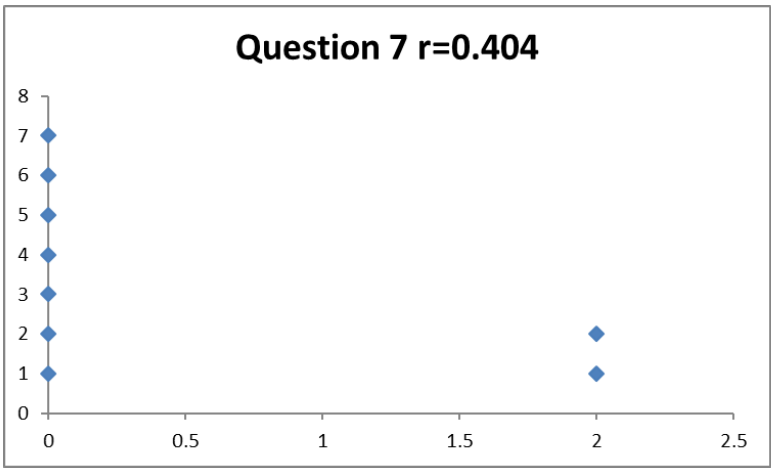
[relevant work]

$x \equiv_5 3$



7) Solve $330x \equiv 1 \bmod 331$ (2 bonus points)

$330x \equiv 1$
$-x \equiv 1$
$x \equiv -1$
$x \equiv 330$



Part 3: Proofs (10 points each, 60 points total)

8) Prove that multiplication in $\mathbb{Z}_n$ is well defined.

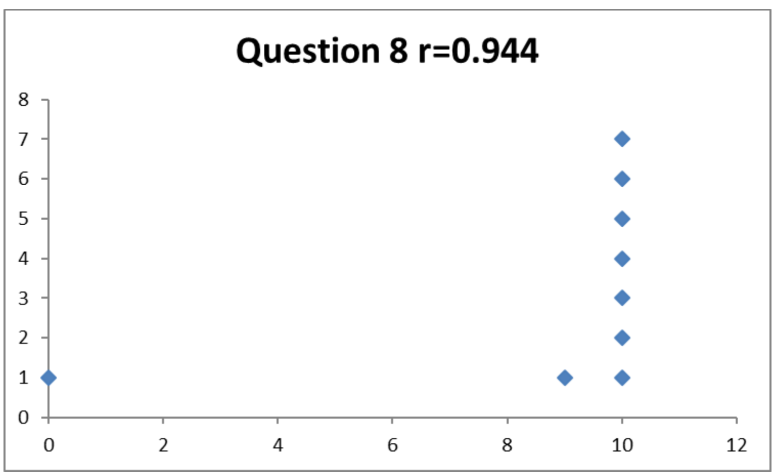
All equivalences in this proof will be mod n.

Let $a_1 \equiv a_2$ and $b_1 \equiv b_2$.

$\therefore n | a_1 - a_2$ and $n | b_1 - b_2$
$\therefore a_1 - a_2 = nk_a$ and $b_1 - b_2 = nk_b$ for some integers k_a and k_b.

$\therefore (a_1 - a_2)(b_1 - b_2) = nk_a - nk_b$
$\therefore a_1b_1 - a_2b_1 - a_1b_2 + a_2b_2 = nk_a - nk_b$
$\therefore a_1b_1 = nk_a - nk_b + a_2b_1 + a_1b_2 - a_2b_2$
$\therefore a_1b_1 - a_2b_2 = nk_a - nk_b + a_2b_1 + a_1b_2 - 2a_2b_2$
$\therefore a_1b_1 - a_2b_2 = nk_a - nk_b + (a_2b_1 - a_2b_2) + (a_1b_2 - a_2b_2)$
$\therefore a_1b_1 - a_2b_2 = nk_a - nk_b + (a_2nk_b) + (nk_ab_2)$
$\therefore n | a_1b_1 - a_2b_2$
$\therefore a_1b_1 \equiv a_2b_2$

This isn't the easiest or cleanest proof. But it's a little more … pedagogical … than what is cleaner.

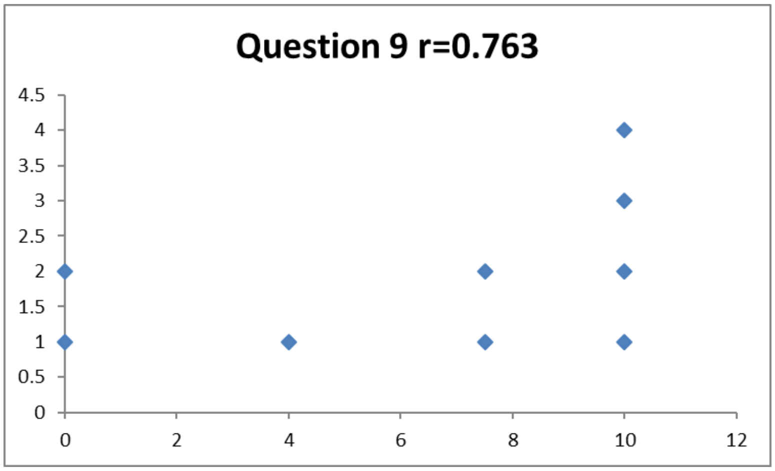


9) Prove the following:

$$1 \in \bigcup_{k=1}^{\infty} \left(\frac{1}{k}, 10 - \frac{1}{k}\right)$$

This is obvious by pulling out the $k = 2$ term:

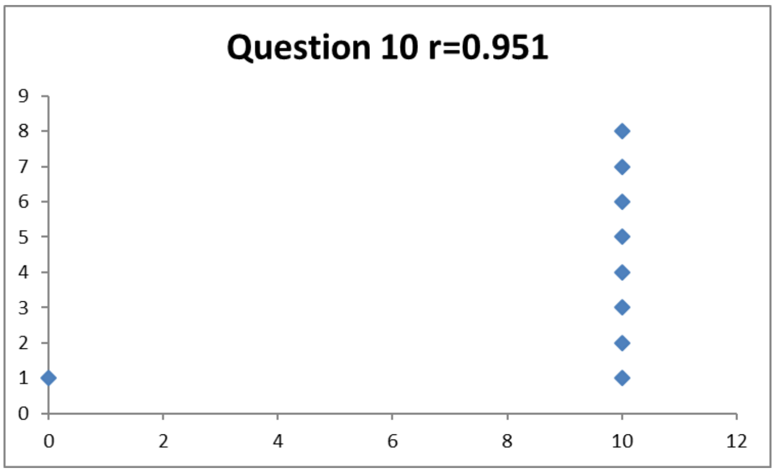
$$1 \in \left(\frac{1}{2}, 9.5\right) \subseteq \bigcup_{k=1}^{\infty} \left(\frac{1}{k}, 10 - \frac{1}{k}\right)$$



10) Prove the function below is injective.

$$f: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto 8x + 2$$

Let $a, b \in \mathbb{R}$
Assume $f(a) = f(b)$
$\therefore 8a + 2 = 8b + 2$
$\therefore 8a = 8b$
$\therefore a = b$
Therefore f is injective.



11) Prove the function below is surjective.

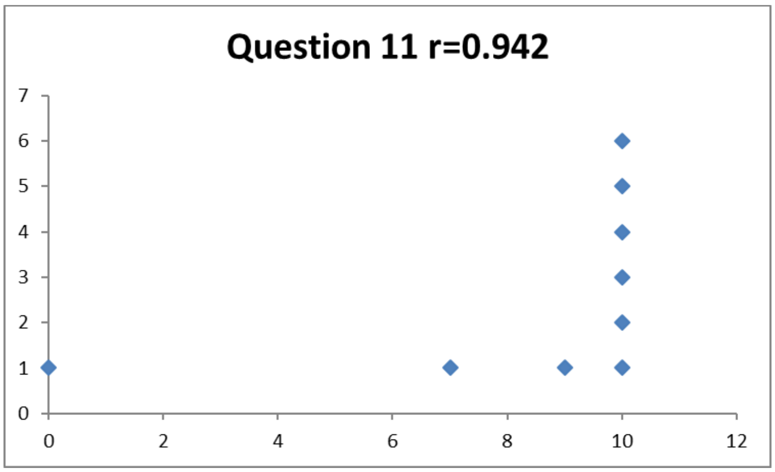
$$f: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto 6x + 5$$

Assume $b \in \mathbb{R}$

Choose $a = \frac{b-5}{6}$

$\therefore f(a) = f\left(\frac{b-5}{6}\right) = 6\left(\frac{b-5}{6}\right) + 5 = b - 5 + 5 = b$

Therefore f is surjective.



12) Prove the inequality below for all integers $n \geq 7$.

$$3^n < n!$$

Base case ($n = 7$):

$$3^7 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \leq 3 \cdot 3 \cdot 27 \cdot 9 \leq 3 \cdot 4 \cdot 30 \cdot 14 = 3 \cdot 4 \cdot (5 \cdot 6) \cdot (7 \cdot 2) = 7!$$

Inductive Step:

Assume $3^k < k!$ For some integer $k \geq 7$.

$\therefore 3^{k+1} = 3 \cdot 3^k < 3 \cdot k! < kk! < (k+1)k! = (k+1)!$

$\therefore 3^{k+1} < (k+1)!$

$\therefore 3^n < n!$ For all integers $n \geq 7$

A note on the base case:

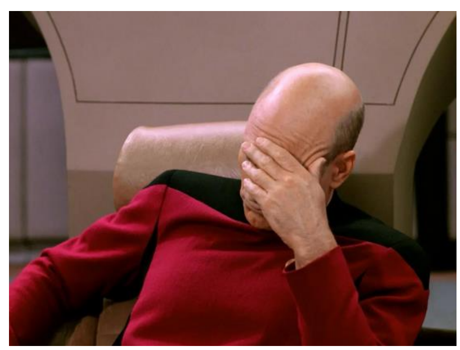
Hopefully it's obvious I didn't expect you to calculate 3^7, you don't have a calculator. If you don't see a clever way to do it like I did, there's another approach. Remember the goal on a test is to communicate what you know, this is probably how I would have solved it in a testing environment:

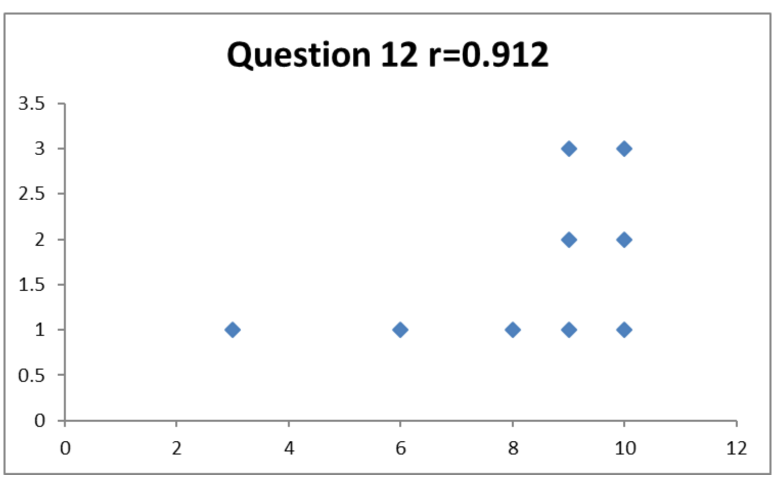
If $n \geq 9$ it's obvious because each 3 can be replaced:

$$3^9 < 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 < 9!$$

For $n = 7$ and $n = 8$ it can be solved easily on a calculator.

SERIOUSLY – SO MANY PEOPLE COMPUTED 3^7 and 7! WHY??
DON'T DO THAT, IT TAKES UP TOO MUCH TIME ON A TEST!!!!





13) Prove the equality below for all integers $n \geq 1$.

$$\sum_{m=1}^{n} (-1)^{m+1} m^2 = \frac{(-1)^{n+1} n(n+1)}{2}$$

Base Case ($n = 1$):

$\sum_{m=1}^{1} (-1)^{m+1} m^2 = (-1)^2 1^2 = 1$

$\frac{(-1)^{1+1} \cdot 1 \cdot (1+1)}{2} = \frac{2}{2} = 1$

$\therefore \sum_{m=1}^{1} (-1)^{m+1} m^2 = \frac{(-1)^{1+1} \cdot 1 \cdot (1+1)}{2}$

Inductive Step:

Assume $\sum_{m=1}^{k} (-1)^{m+1} m^2 = \frac{(-1)^{k+1} k(k+1)}{2}$ for some $k \in \mathbb{N}$.

$$\therefore \sum_{m=1}^{k+1} (-1)^{m+1} m^2 = \sum_{m=1}^{k} (-1)^{m+1} m^2 + (-1)^{k+2}(k+1)^2 = \frac{(-1)^{k+1} k(k+1)}{2} + (-1)^{k+2}(k+1)^2$$

$$= (-1)^{k+2}\left[\frac{-k(k+1)}{2} + \frac{2(k+1)^2}{2}\right]$$

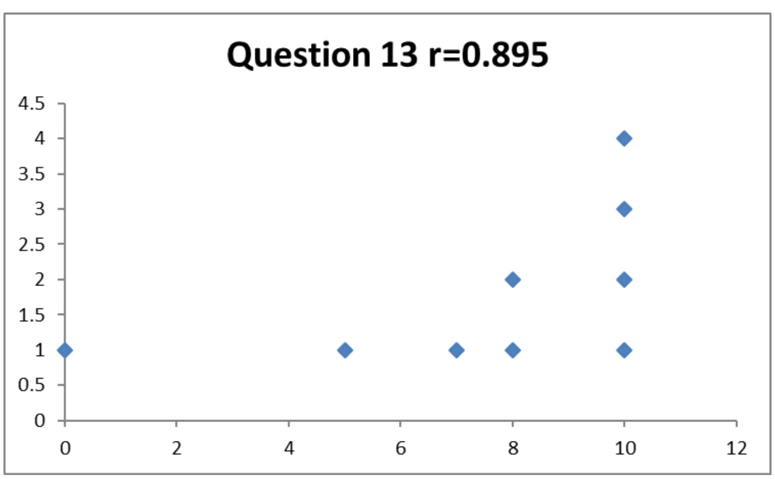
$$= (-1)^{k+2}(k+1)\left[\frac{-k}{2} + \frac{2(k+1)}{2}\right]$$

$$= (-1)^{k+2}(k+1)\left[\frac{-k+2k+2}{2}\right]$$

$$= \frac{(-1)^{k+2}(k+1)(k+2)}{2}$$

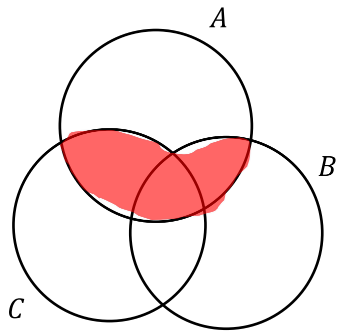
Therefore by induction $\sum_{m=1}^{n} (-1)^{m+1} m^2 = \frac{(-1)^{n+1} n(n+1)}{2}$ for all $n \in \mathbb{N}$

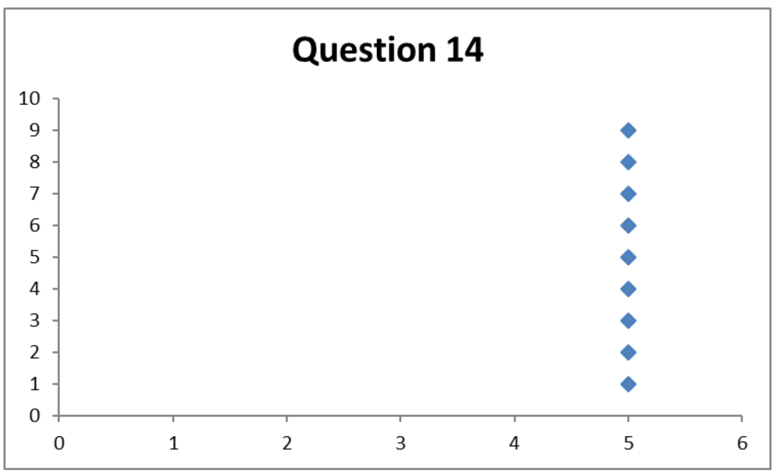
Grading note: I intended/expected this to be the hardest proof. Once you get into it I don't think it's that hard, but it has an intimidation factor: especially considering the homework didn't have any exercises with a $(-1)^m$ term. This was probably as "outside the box" as an equality induction problem can get.



Part 4: Review

14) Let A, B, and C be sets. Draw a Venn Diagram to illustrate $A \cap (B \cup C)$. (5 points)





15) What is the term used to describe the mistake when a proof writer assumes the conclusion they're trying to prove? (5 points)

(A) Conclusion Reasoning
(B) Concussion Reasoning
(C) Circular Reasoning
(D) Implication Reasoning
(E) Wrap-around Reasoning