Name ____ Test 2, Spring 2022

Part 1: Basic Knowledge (5 points each, 20 points total)

For each problem, give a precise definition.

1) Consider the function notation below. What is "$x \mapsto f(x)$" called?

$$\begin{aligned} f: A &\to B \\ x &\mapsto f(x) \end{aligned}$$

The rule.

2) Let S be a set. What is the definition of a relation on S?

Any subset of $S \times S$.

For comparison, a relation from A to B is any subset of $A \times B$.

3) Let $f: A \to B$ be a function. What does it mean for f to be onto? Give the definition.

$$\forall_{b \in B} \exists_{a \in A} (f(a) = b)$$

Or in words, "for everything in the codomain, there is something in the domain that maps to it"

Or a little more conceptually, "the range of f is the entire codomain of f"

4) Let R be a relation on the set S. What does it mean for R to be symmetric? Give the definition.

For all $x, y \in S$, if xRy, then yRx.

Or

$$\forall_{x,y \in S} (xRy \to yRx)$$

(No points were deducted if you forgot the universal… but you shouldn't forget it!)

Note the key difference between implication and conjunction. IF xRy, then yRx is different from saying BOTH xRy and yRx.

Part 2: Basic Skills and Concepts (5 points each, 20 points total)

5) Find the indexed union below.

$$\bigcup_{k=1}^{\infty}\left[\frac{1}{k}, 2+\frac{1}{k}\right]$$

$$(0,3]$$

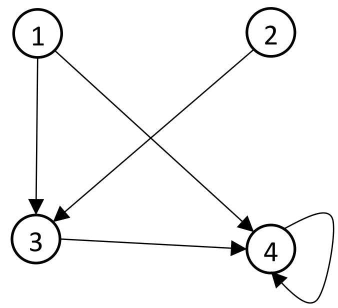
6) Evaluate the function below at 5.

$$f\colon \mathbb{R} \to \mathbb{R}^2$$
$$x \mapsto (3x+2, x^2)$$

$f(5) = (3 \cdot 5 + 2, 5^2) = (17, 25)$

7) Illustrate the relation below as a digraph. (The one with circular nodes and arrows between them)

$$\{(1,3),(1,4),(2,3),(3,4),(4,4)\}$$



8) Which of the following relations are symmetric? Circle Y for Yes and N or No.

Y N a) R on $\mathbb{R}$ given by xRy iff $xy = 5$.

Y N b) R on the set of polynomials given by fRg iff $f(0) \leq g(0)$

Y N c) R on $\mathbb{Z}$ given by xRy iff $x - 2y = 0$

Y N d) R on $\mathbb{Z}$ given by xRy iff $5|x - y$

Y N e) R on the set of people given by pRq iff p and q have the same last name.

Part 3: Proofs (10 points each, 40 points total)

9) Consider the relation below. Prove that it is reflexive.

Let S be the set of all polynomials. Define R as the relation on S given by:

$$fRg \text{ iff } \deg(f) \leq \deg(g)$$

Let f be a polynomial. Then $\deg(f) = \deg(f)$, so $\deg(f) \leq \deg(f)$. Therefore fRf.

10) Prove that the function below is injective (also known as one-to-one):

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto 14x - 3$$

Let $x_1, x_2 \in \mathbb{R}$	
Assume $f(x_1) = f(x_2)$	
$\therefore 14x_1 - 3 = 14x_2 - 3$	Plug in f
$\therefore 14x_1 = 14x_2$	Add 3 to both side
$\therefore x_1 = x_2$	Divide by 14 on both sides
Therefore f is injective.	Definition of injective.

11) Let I be an index set and A_k a set for each $k \in I$. Prove that:

$$\forall_{j \in I} \left(A_j \subseteq \bigcup_{i \in I} A_i \right)$$

Let $j \in I$.

Assume $x \in A_j$

$\therefore x \in \bigcup_{i \in I} A_i$ Definition of union

$\therefore A_j \subseteq \bigcup_{i \in I} A_i$ Definition of subset

Therefore for all $j \in I$, $A_j \subseteq \bigcup_{i \in I} A_i$. Universal generalization

12) Prove that the function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$ is invertible.

We show that it is invertible by proving that $g(x) = \frac{1}{2}x$ is the inverse:

$$(f \circ g)(x) = f\big(g(x)\big) = 2\left(\frac{1}{2}x\right) = x$$

$$(g \circ f)(x) = g\big(f(x)\big) = \frac{1}{2}(2x) = x$$

Part 4: Review (20 points total)

13) Determine whether these statements are true or false.
(5 points)

T F a) $\forall_{x \in \mathbb{R}}(x^2 \geq 4)$
T F b) $\forall_{x \in \mathbb{R}} \exists_{y \in \mathbb{R}}(x + y = 4)$
T F c) $\exists_{x \in \mathbb{R}} \forall_{y \in \mathbb{R}}(xy = 0)$
T F d) $\exists_{x \in \mathbb{Z}}(6|x)$
T F e) $\forall_{x \in \mathbb{R}} \forall_{y \in \mathbb{R}} \forall_{z \in \mathbb{R}}(xyz = 1)$

14) Find $[3,5] \cap (4,6)$.
(5 points)

$(4,5]$

15) If x is an even integer and y is an odd integer, prove that $2x + y$ is odd.
(10 points)

Let x be even and y an odd integer.

$\therefore x = 2k$ for some $k \in \mathbb{Z}$	Definition of even
$\therefore y = 2l + 1$ for some $l \in \mathbb{Z}$	Definition of odd
$\therefore 2x + y = 4k + 2l + 1 = 2(2k + l) + 1$	Algebra
$\therefore 2x + y$ is odd	Definition of odd