Name ____________________________________________________________ Test 3, Spring 2022

**Part 1: Basic Knowledge** (5 points each, 10 points total)

1) What does it mean for $a \equiv_n b$? State the definition.

$$n|b - a$$

2) What does it mean addition to be <u>well defined</u> mod $n$? State the definition.

If $a_1 \equiv_n a_2$ and $b_1 \equiv_n b_2$, then $a_1 + b_1 \equiv_n a_2 + b_2$.

Or equivalently,

If $[a_1]_n = [a_2]_n$ and $[b_1]_n = [b_2]_n$, then $[a_1 + b_1]_n = [a_2 + b_2]_n$.

**Part 2: Basic Skills and Concepts** (5 points each, 20 points total)

3) Find $3 \cdot 6 - 4 \bmod 10$.

$18 - 4 \equiv 14 \equiv 4$

4) Solve $3x \equiv 5 \bmod 7$.

$x \equiv 4$

5) Solve $3x \equiv 6 \bmod 12$.

$x \equiv 2,6,10$

6) What is $[5]_{20}$? No words please. Just math.

$[5]_{20} = \{\cdots, 5, 25, 45, \cdots\}$

**Part 3: Proofs** (50 points total)

7) Prove that $[5]_{10} \cap [6]_{10} = \emptyset$

(It is not enough to write them down and point to it, though that would get you partial credit. *Prove* that they have nothing in common, please!) (10 points)

Suppose $x \in [5]_{10} \cap [6]_{10}$.

$\therefore x \in [5]_{10}$

$\therefore x = 10k + 5$ for some $k \in \mathbb{Z}$

$\therefore x \in [6]_{10}$

$\therefore x = 10l + 6$ for some $l \in \mathbb{Z}$

$\therefore 10k + 5 = 10l + 6$

$\therefore 10k - 10l = 6 - 5 = 1$

$\therefore 10|1$

This is a contradiction. Therefore $x \notin [5]_{10} \cap [6]_{10}$

Because $x$ was arbitrary, $[5]_{10} \cap [6]_{10} = \emptyset$

OR

$[5]_{10}$ and $[6]_{10}$ are both equivalence classes mod 10. By previous theorem, equivalence classes form a partition, which are inherently disjoint. Therefore $[5]_{10} \cap [6]_{10} = \emptyset$.

8) Prove the equality below for all integers $n \geq 1$. (20 points)

$$\sum_{l=1}^{n} \frac{1}{(2l-1)(2l+1)} = \frac{n}{2n+1}$$

Base Case:

$$\sum_{l=1}^{1} \frac{1}{(2l-1)(2l+1)} = \frac{1}{(1)(3)} = \frac{1}{3}$$

$$\frac{1}{2 \cdot 1 + 1} = \frac{1}{3}$$

$$\therefore \sum_{l=1}^{1} \frac{1}{(2l-1)(2l+1)} = \frac{1}{2 \cdot 1 + 1}$$

Assume $\sum_{l=1}^{k} \frac{1}{(2l-1)(2l+1)} = \frac{k}{2k+1}$ for some $k \in \mathbb{Z}$

$$\sum_{l=1}^{k+1} \frac{1}{(2l-1)(2l+1)} = \sum_{l=1}^{k} \frac{1}{(2l-1)(2l+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k(2k+3)}{(2k+1)(2k+3)} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$= \frac{(k+1)}{(2k+3)}$$

$$= \frac{(k+1)}{(2(k+1)+1)}$$

Therefore by induction, $\sum_{l=1}^{n} \frac{1}{(2l-1)(2l+1)} = \frac{n}{2n+1}$ for all $n \in \mathbb{N}$.

9) Prove the inequality below for all integers $n \geq 2$. (20 points)

$$n! < n^n$$

Base Case:

$$2! = 2 < 4 = 2^2$$

Assume $k! < k^k$ for some integer $k \geq 2$.

$$(k+1)! = (k+1)k! < (k+1)k^k < (k+1) \cdot (k+1)^k = (k+1)^{k+1}$$

Therefore $n! < n^n$ for all $n \in \mathbb{N}$.

**Part 4: Review** (20 points total)

10) Find $\{3,4,5,6,7\} - \{2,3,4,5\}$
(5 points)

$\{6,7\}$

11) What is the truth table for $P \Rightarrow Q$?
(5 points)

| $P$ | $Q$ | $P \Rightarrow Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

12) Prove that if $x$ and $y$ are both rational, then $x + y$ is rational.
(10 points)

Assume $x, y \in \mathbb{Q}$. Then $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for some $a, b, c, d \in \mathbb{Z}$. Therefore:

$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad + cb}{bd} \in \mathbb{Q}$$