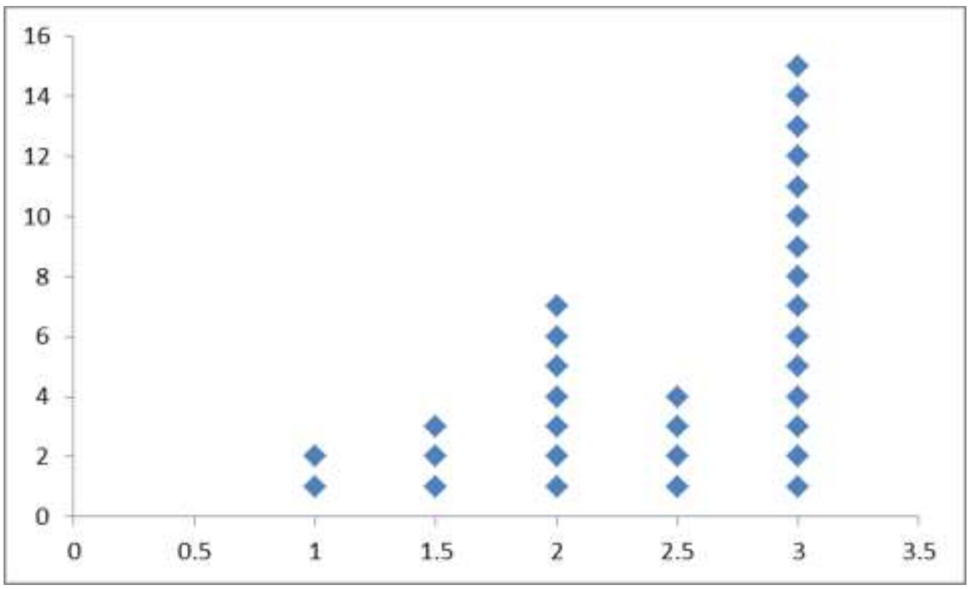
Name__Solutions____ Linear Algebra; Test 1

1) Given the system of equations below, describe the solution set. (Don't find it, just describe it in one English sentence or mathematical expression) (3 points)

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 5 \\ x_2 + x_3 + x_4 &= 2 \\ x_3 + x_4 &= 7 \end{aligned}$$

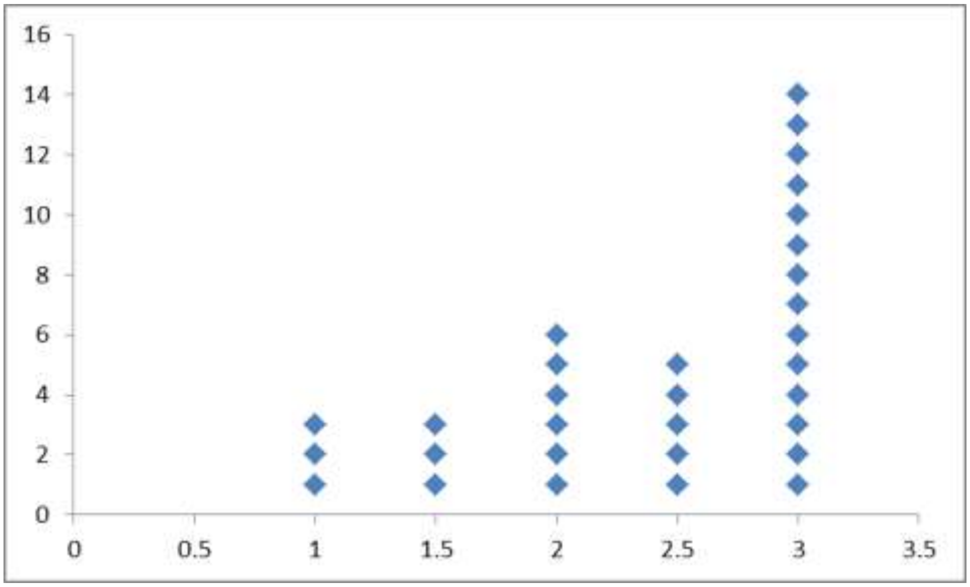
With exactly one free variable, this is a $\mathbb{R}^1$ space.



2) Given the system of equations below, describe the solution set. (Don't find it, just describe it in one English sentence or mathematical expression) (3 points)

$$x_1 + x_2 + x_3 + x_4 = 5$$
$$x_2 + x_3 + x_4 = 2$$

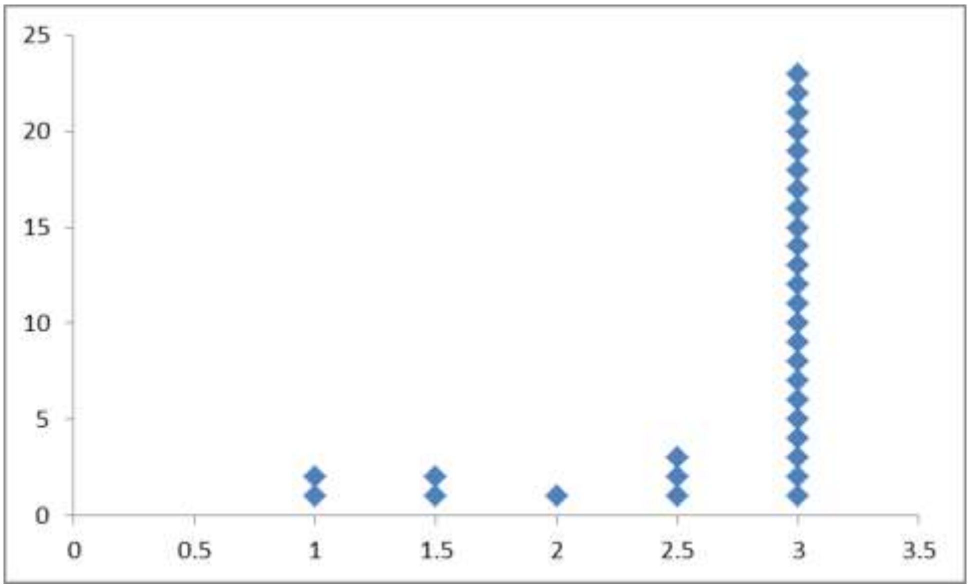
With exactly two free variables, this is a $\mathbb{R}^2$ space.



3) Given the system of equations below, describe the solution set. (Don't find it, just describe it in one English sentence or mathematical expression) (3 points)

$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ x_2 + x_3 &= 2 \\ x_3 &= 7 \end{aligned}$$

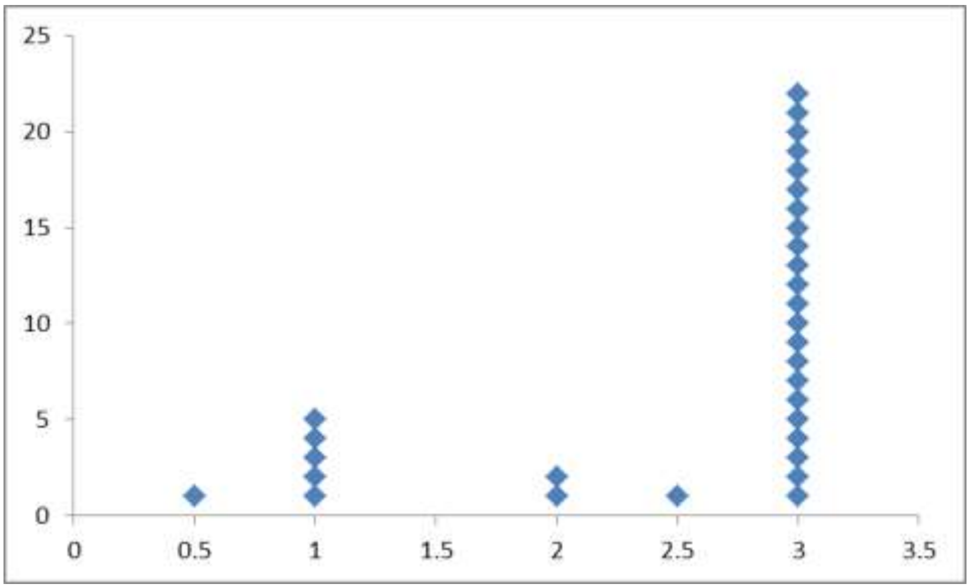
With no free variables, this system has a unique solution.



4) Is $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ a solution to the matrix equation $\begin{bmatrix}2 & 3 & 4 & 5\\1 & 1 & 1 & 1\end{bmatrix}\vec{x} = \begin{bmatrix}10\\5\end{bmatrix}$? Why or why not? (3 points)

No:

$$\begin{bmatrix}2 & 3 & 4 & 5\\1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}2+3+4+5\\1+1+1+1\end{bmatrix} = \begin{bmatrix}14\\4\end{bmatrix} \neq \begin{bmatrix}10\\5\end{bmatrix}$$



5) Define the function $T: \mathbb{R}^2 \to \mathbb{R}^2$ via $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 0 \end{bmatrix}$. Explain or show why T is a linear transformation. (8 points)

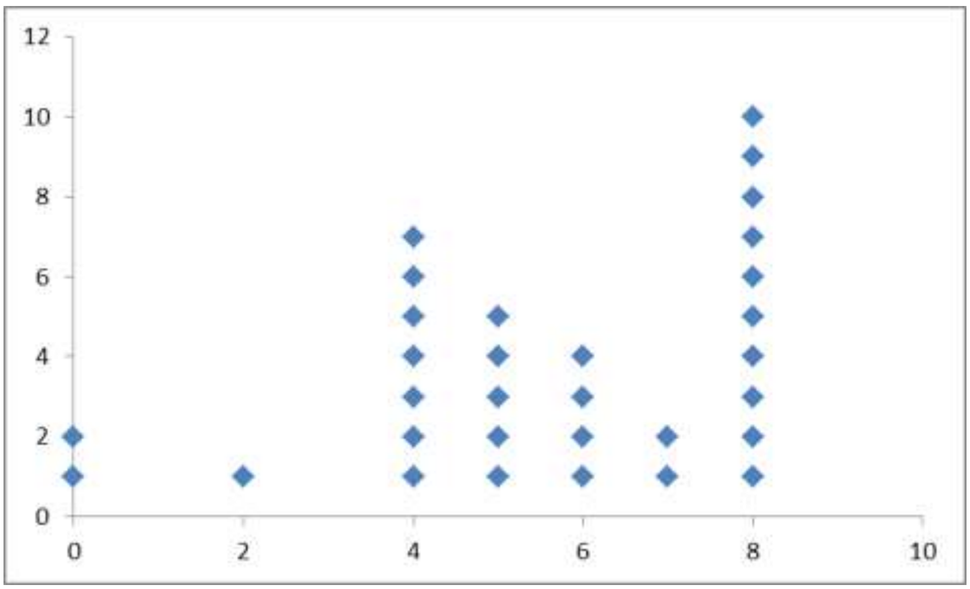
"Easy" way: (using a theorem from class)

T is associated to the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and any such function is a linear transformation.

"Hard" way: (using a definition from class)

$$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + y_1 + y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 + y_2 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

$$T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = \begin{bmatrix} cx + cy \\ 0 \end{bmatrix} = c\begin{bmatrix} x + y \\ 0 \end{bmatrix} = cT\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right)$$



6) Let T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Place each of the following terms in the circles below such that all items in a circle are equivalent. Note that not all circles may be used. (20 points)

a) T is one-to-one.

b) T is onto.

c) The range of T is $\mathbb{R}^m$.

d) The columns of T are linearly independent.

e) The columns of T are linearly dependent.

f) The columns of T span $\mathbb{R}^m$

g) $[T]$ has full column span.

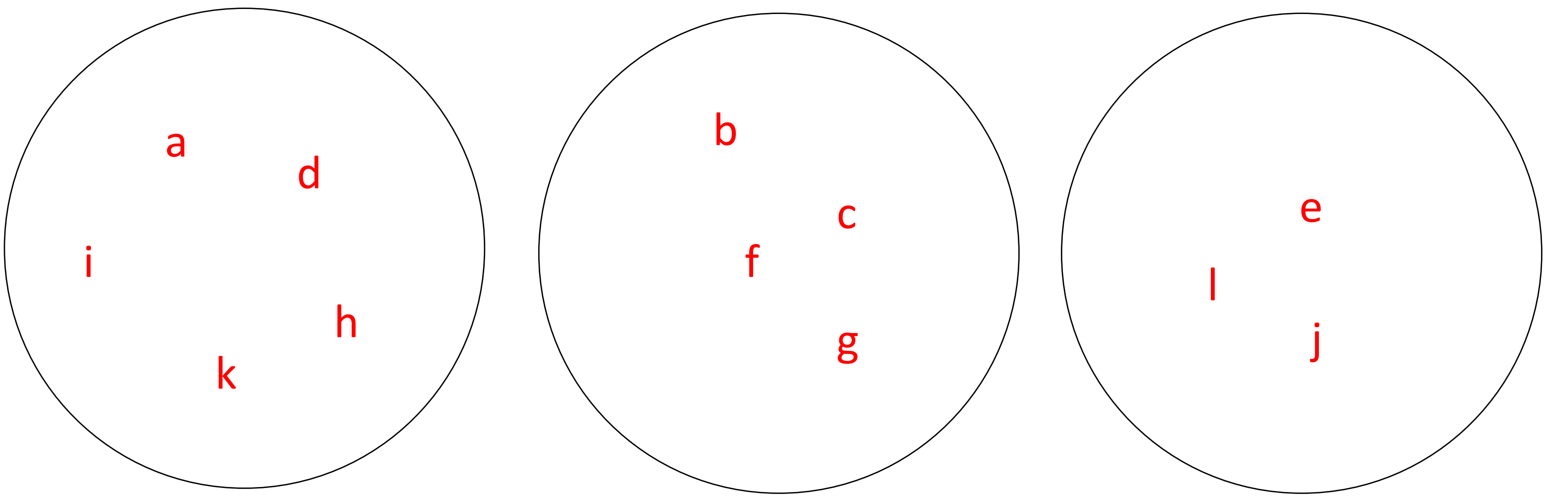
h) $[T]$'s echelon form shows that the corresponding equation has no free variables.

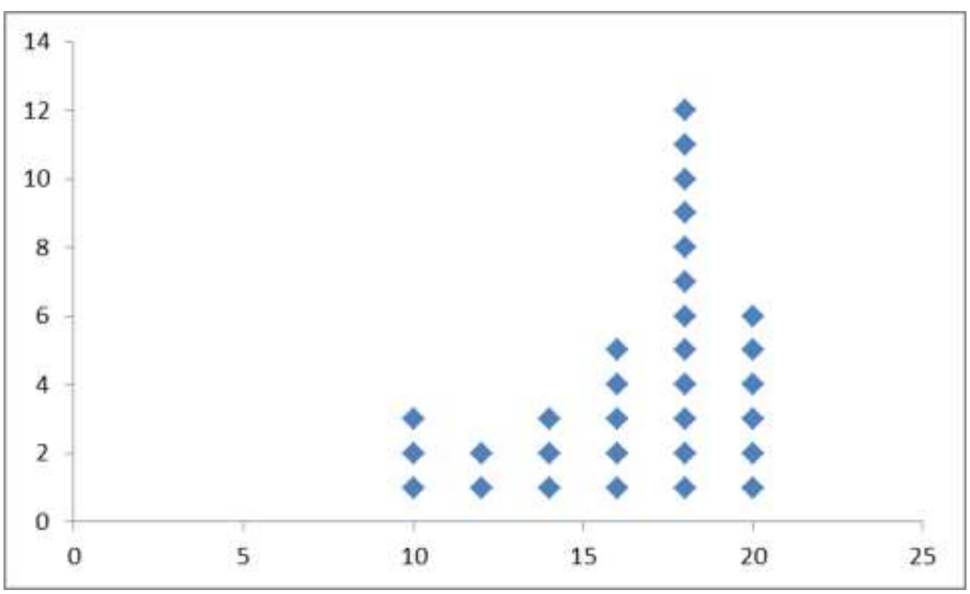
i) $[T]\vec{x} = \vec{0}$ has exactly one solution.

j) $[T]\vec{x} = \vec{0}$ has multiple solutions.

k) $[T]\vec{x} = \vec{b}$ has at most one solution.

l) $[T]\vec{x} = \vec{b}$ has multiple solutions.

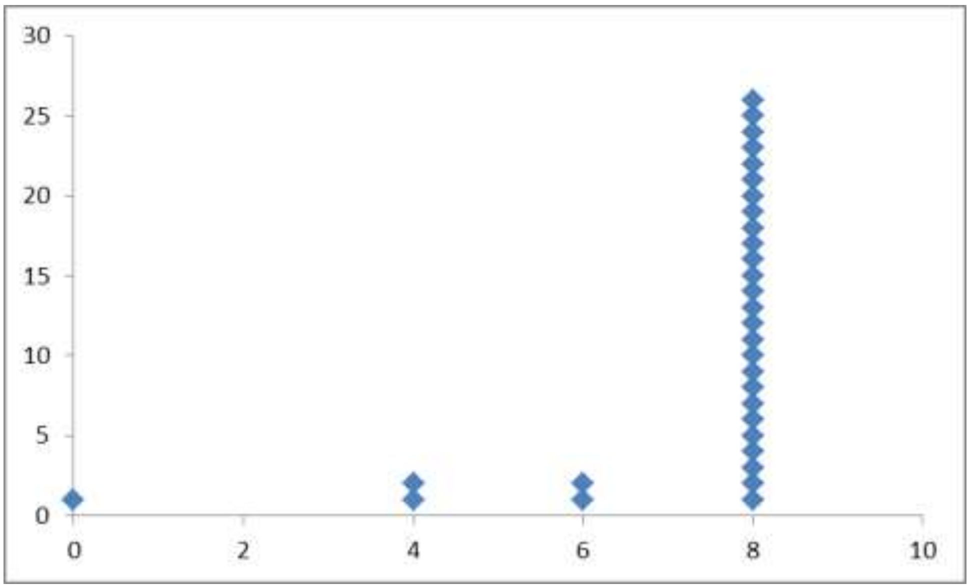




7) Find a matrix in row echelon form that is row equivalent to the matrix below. (8 points)

$$\begin{bmatrix} 2 & 0 & 1 \\ 4 & 1 & 2 \\ 8 & 4 & 4 \end{bmatrix}$$

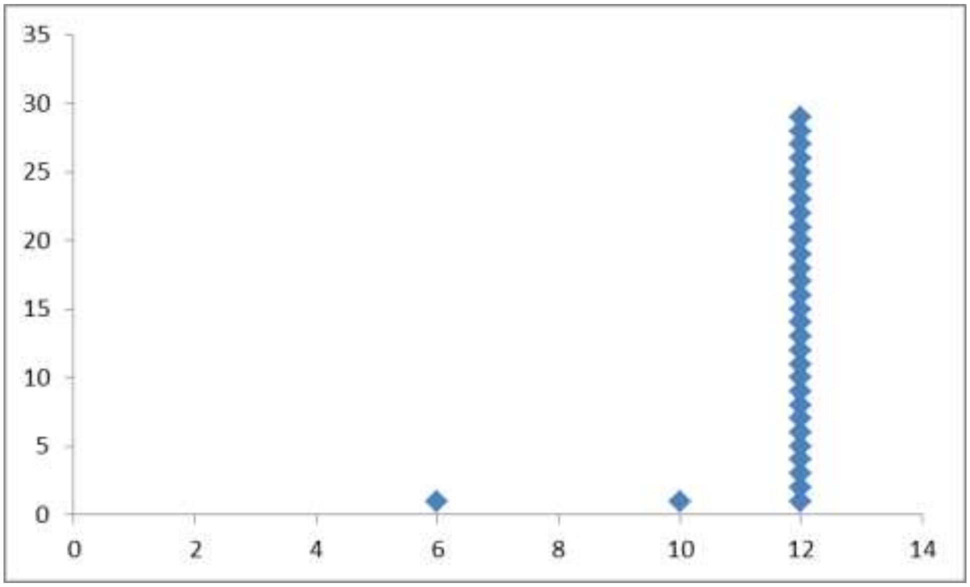
$$\begin{bmatrix} 2 & 0 & 1 \\ 4 & 1 & 2 \\ 8 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 8 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



8) Find a matrix in reduced row echelon form that is row equivalent to the matrix below. (12 points)

$$\begin{bmatrix} 2 & 0 & 1 \\ 4 & 1 & 2 \\ 8 & 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 4 & 1 & 2 \\ 8 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



9) Find the inverse of the matrix below. (10 points)

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 9 & 0 \\ 5 & 0 & 0 & 6 \end{bmatrix}$$

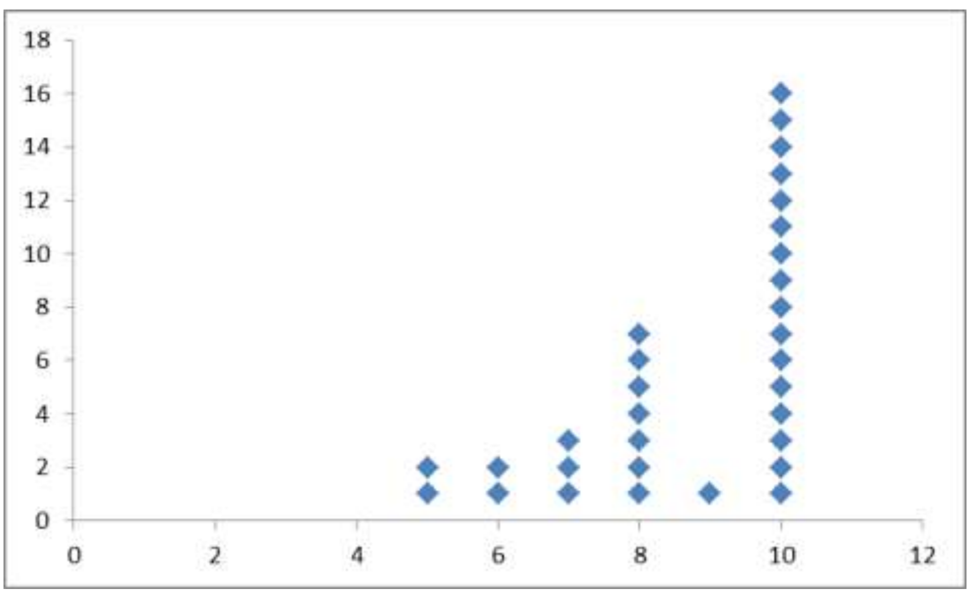
$$\begin{bmatrix} 1 & 0 & 0 & 1 & | & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 0 & | & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 6 & | & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & | & -2 & 1 & 0 & 0 \\ 0 & 0 & 9 & 0 & | & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 6 & | & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & | & -2 & 1 & 0 & 0 \\ 0 & 0 & 9 & 0 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & | & -5 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & | & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & | & 0 & 0 & 1/9 & 0 \\ 0 & 0 & 0 & 1 & | & -5 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & | & -7 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & | & 0 & 0 & 1/9 & 0 \\ 0 & 0 & 0 & 1 & | & -5 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & | & 6 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & | & -7 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & | & 0 & 0 & 1/9 & 0 \\ 0 & 0 & 0 & 1 & | & -5 & 0 & 0 & 1 \end{bmatrix}$$

Thus the inverse matrix is:

$$\begin{bmatrix} 6 & 0 & 0 & -1 \\ -7 & 1 & 0 & 1 \\ 0 & 0 & 1/9 & 0 \\ -5 & 0 & 0 & 1 \end{bmatrix}$$



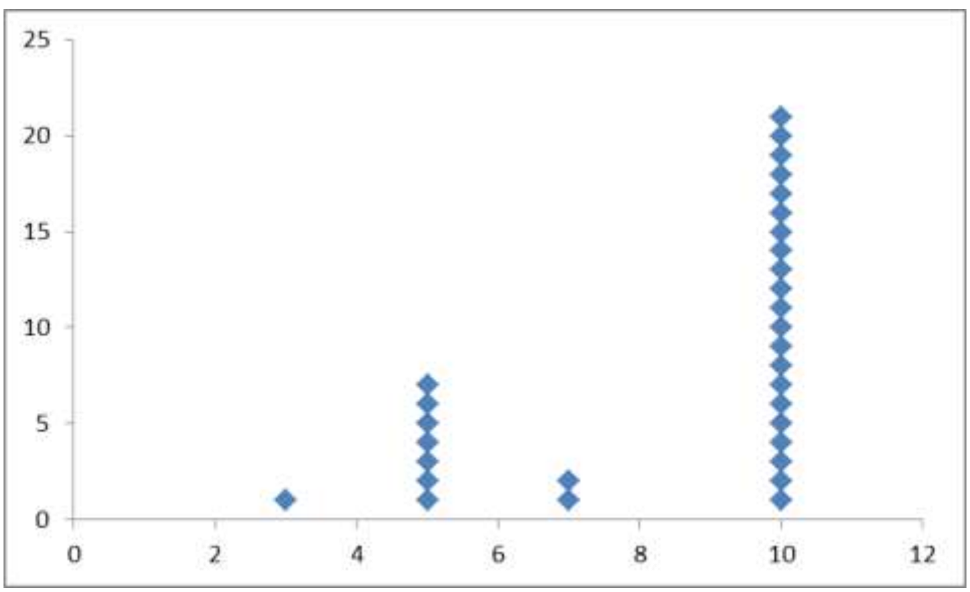
10) Solve for $\vec{x}$. (10 points)

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 9 & 0 \\ 5 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 & 0 & -1 \\ -7 & 1 & 0 & 1 \\ 0 & 0 & 1/9 & 0 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 9 & 0 \\ 5 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 & -1 \\ -7 & 1 & 0 & 1 \\ 0 & 0 & 1/9 & 0 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0+0+0+0 \\ 0+1+0+0 \\ 0+0+2/9+0 \\ 0+0+0+0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2/9 \\ 0 \end{bmatrix}$$

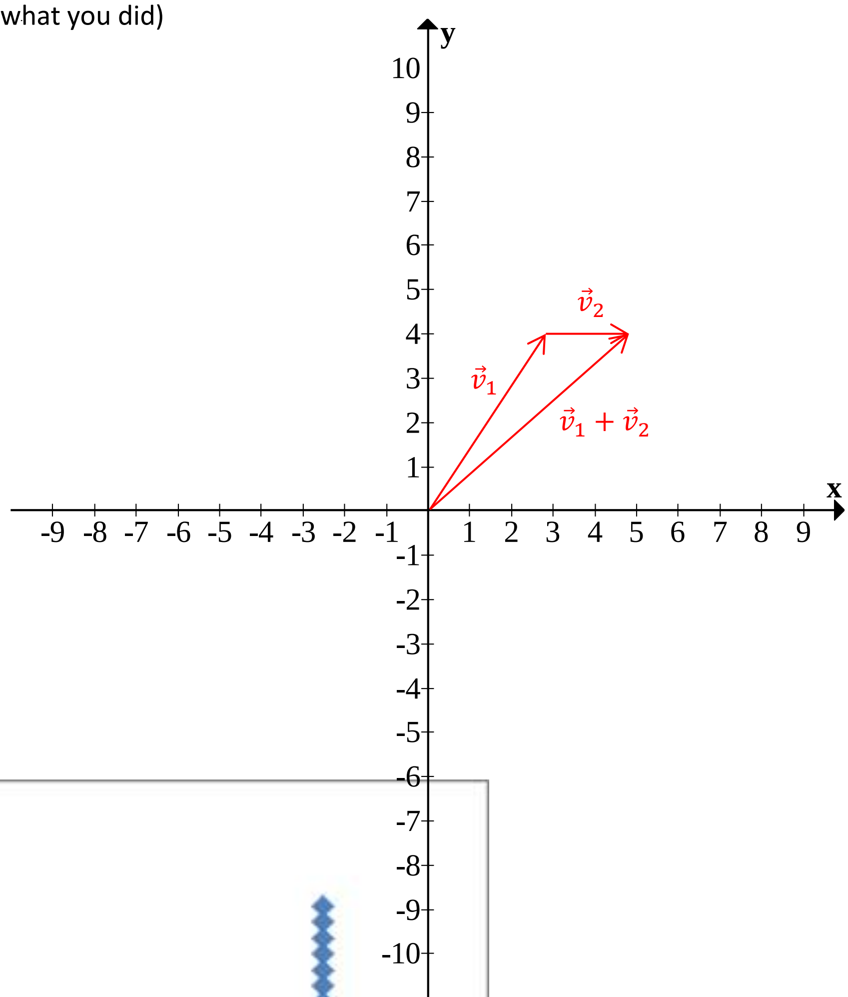


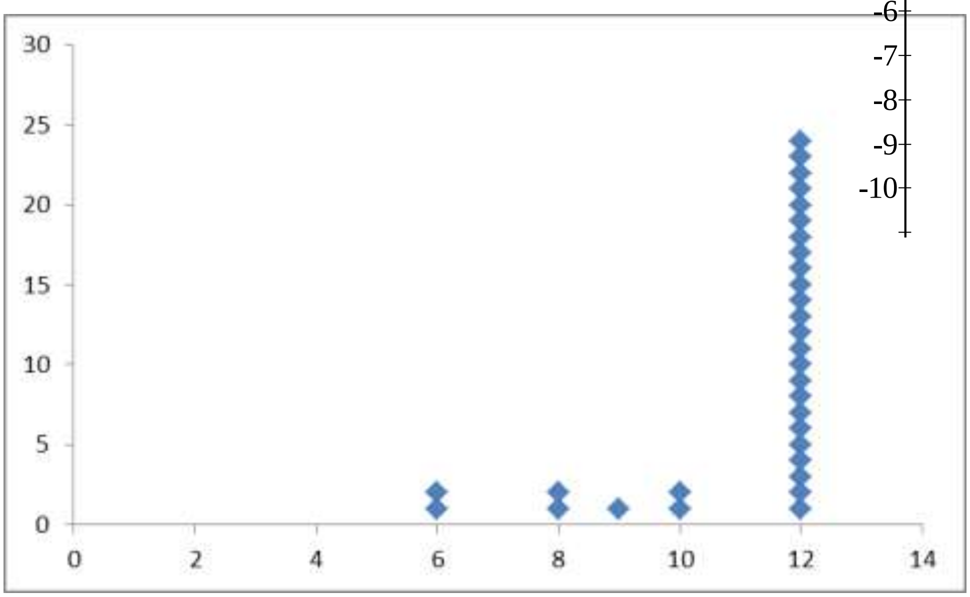
11) On the graph below, sketch and label the two follow vectors below. (12 points)

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

(For $\vec{v}_2$, find it graphically and illustrate what you did)





12) Which of the following collections of vectors are linearly dependent? Circle them. (8 points)

$$\left\{\begin{bmatrix}2\\3\\5\end{bmatrix}, \begin{bmatrix}4\\6\\10\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$$

$$\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\4\end{bmatrix}, \begin{bmatrix}7\\2\\3\end{bmatrix}, \begin{bmatrix}9\\2\\3\end{bmatrix}\right\}$$

$$\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$

$$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}\right\}$$