1) Give a matrix with the following column space: (5 points)

\[ \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right) \]

There are many possible answers, the simplest is probably:

\[
\begin{bmatrix}
1 & 0 \\
6 & 0 \\
2 & 1
\end{bmatrix}
\]
2) Give a linear operator whose associated matrix has the following row space: (5 points)

\[
\text{span}\left(\left\{ \begin{bmatrix} 1 \\ 6 \\ 2 \\ 0 \\ 1 \end{bmatrix}^t, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}^t \right\} \right)
\]

There are many possible answers. The simplest is probably the linear operator corresponding to \[
\begin{bmatrix} 1 & 6 & 2 \\ 0 & 0 & 1 \end{bmatrix}.
\]

\[
T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 6x_2 + 2x_3 \\ x_3 \end{bmatrix}
\]
3) Give a matrix with the following null space: (5 points)

\[
\text{span}\left(\begin{bmatrix}
1 \\
6 \\
2 \\
1
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}\right)
\]

There are many possible answers. The simplest is probably:

\[
\begin{bmatrix}
6 & -1 & 0
\end{bmatrix}
\]
4) Give an example of a homogeneous system of equations in which the associated linear transformation has nontrivial kernel. (5 points)

There are many possible answers. One simple such answer is:

\[ x_1 + x_2 = 0 \]
5) Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a linear operator with trivial kernel. Prove that the columns of $[T]$ are linearly independent. (10 points)

Denote the columns of $[T]$ as $\vec{v}_1, \ldots, \vec{v}_n$. Assume $a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n = \vec{0}$. Then as a matrix equation this means $[T] \vec{a} = \vec{0}$ where $\vec{a} = [a_1 \ a_2 \ \cdots \ a_n]^T$. Because the kernel is trivial, $\vec{a} = \vec{0}$. That is to say, $a_1 = 0, a_2 = 0, \ldots, a_n = 0$. Hence $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent. □

Grading:
10 points if it looks like you know what you’re doing.
8 points if you seem to have an idea, but made some glaring mistakes.
5 points if you’re completely off base but made a reasonable attempt.

...a couple people appear to have memorized a proof for something else. Bad!
6) Find a basis for \( \left\{ \begin{pmatrix} 0 \\ 1 \\ a \\ a + b \\ c \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \). (5 points)

There are many possible answers. The simplest is probably:

\[
\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}
\]

Note that technically speaking this set is not a subspace of \( \mathbb{R}^5 \); it was graded according to whether or not your answer was linearly independent and spanned this set.

Note that this problem was re-graded several times. The pink marks are corrections to the red marks, and the orange marks are corrections to the red and/or pink marks.
7) Give an example of a linear operator $T$ such that the associated linear system $[T] \vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^3$. (5 points)

There are many answers. One possible answer is:

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
8) Find the null space of \[
\begin{bmatrix}
1 & -5 \\
-3 & 15 \\
2 & -10
\end{bmatrix},
\] (10 points)

\[
\begin{bmatrix}
1 & -5 \\
-3 & 15 \\
2 & -10
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -5 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

Solving \[
\begin{bmatrix}
1 & -5 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\] we find that \(x_2\) can be free, while \(x_1\) is five times \(x_2\). Hence the null space is:

\[
\text{span}\left(\begin{bmatrix}
5 \\
1
\end{bmatrix}\right)
\]
9) Alice is an aspiring linear algebraist. Find a set of three vectors in $\mathbb{R}^4$ such that when any one is removed, Alice can find two new vectors to add to the set to make a basis for $\mathbb{R}^4$. (5 points)

There are many possible answers. One such answer is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$
10) If $T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 \end{pmatrix}$, find $[T^{-1}]$. That is, find the associated matrix of the inverse function. (5 points)

$[T] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$[T^{-1}] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$
11) A $7 \times 5$ matrix has just 3 linearly dependent rows. How many free variables are there in the associated system of homogeneous equations? (10 points)

This question was thrown out because it is not well defined: It is not clear what “having 3 linearly dependent rows” means. The question was intended to say “A $7 \times 5$ matrix has just 3 linearly independent rows.”

In that case, the nullity of the matrix would be 2, so there would be two free variables.
12) Express 432 different bases for $\mathbb{R}^4$. (Hint: Don't try to write them all down) (5 points)

$$\left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ a \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ a \end{bmatrix} \right\} \text{ for } a = 1, \ldots, 432$$
13) Find two examples of dimension 2 spaces. (5 points)

There are many such spaces. Two of these spaces are:

\( \mathbb{R}^2 \) and \( \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \)

Grading note: Points were deducted if you gave the same space written in two different ways, such as

\( \mathbb{R}^2 \) and \( \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \)
14) The graph below illustrates two vectors, $\vec{v}_1$ and $\vec{v}_2$. The picture illustrates that $T(\vec{v}_1) = \vec{v}_2$. Find the associated matrix $[T]$. (10 points)

We must find a $2 \times 2$ matrix $[T]$ such that:

$$[T] \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

There are many such matrices. One example is $[T] = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -3 \end{bmatrix}$. 
15) Find all values of \( x \) so that \( \text{rank}(A) = 2 \), when \( A = \begin{bmatrix} -1 & 2 & 1 \\ 3 & 1 & 11 \\ 4 & 3 & x \end{bmatrix} \). (10 points)

The first two rows are linearly independent. Hence to be rank 2, the third row must be linearly dependent. Reducing \( A \) we find:

\[
\begin{bmatrix}
-1 & 2 & 1 \\
3 & 1 & 11 \\
4 & 3 & x
\end{bmatrix}
\sim
\begin{bmatrix}
-1 & 2 & 1 \\
0 & 1 & 2 \\
0 & 0 & x - 18
\end{bmatrix}
\]

Hence for this matrix to have rank 2, \( x \) must be 18.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{graph.png}
\caption{Graph showing the solution to the problem.}
\end{figure}