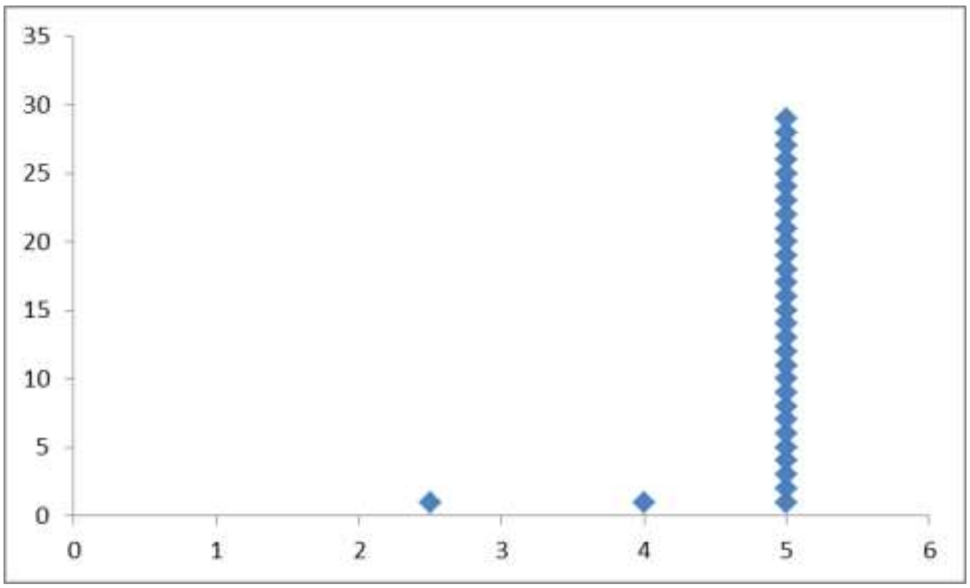
Name___Solutions____ Linear Algebra; Test 2

1) Give a matrix with the following column space: (5 points)

$$\text{span}\left(\left\{\begin{bmatrix}1\\6\\2\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}\right)$$

There are many possible answers, the simplest is probably:

$$\begin{bmatrix}1 & 0\\6 & 0\\2 & 1\end{bmatrix}$$

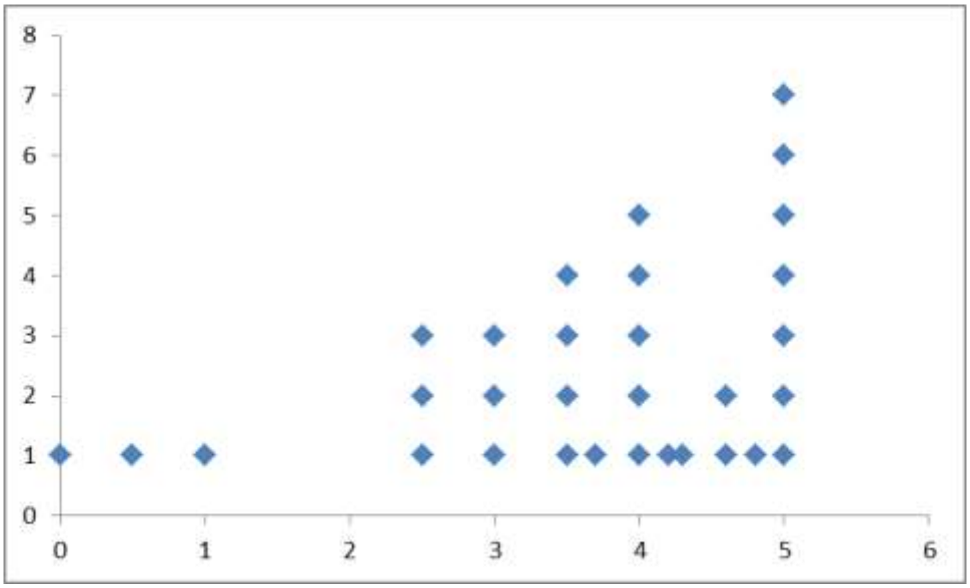


2) Give a linear operator whose associated matrix has the following row space: (5 points)

$$\text{span}\left(\left\{\begin{bmatrix}1\\6\\2\end{bmatrix}^t, \begin{bmatrix}0\\0\\1\end{bmatrix}^t\right\}\right)$$

There are many possible answers. The simplest is probably the linear operator corresponding to $\begin{bmatrix}1 & 6 & 2\\0 & 0 & 1\end{bmatrix}$:

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}x_1 + 6x_2 + 2x_3\\x_3\end{bmatrix}$$

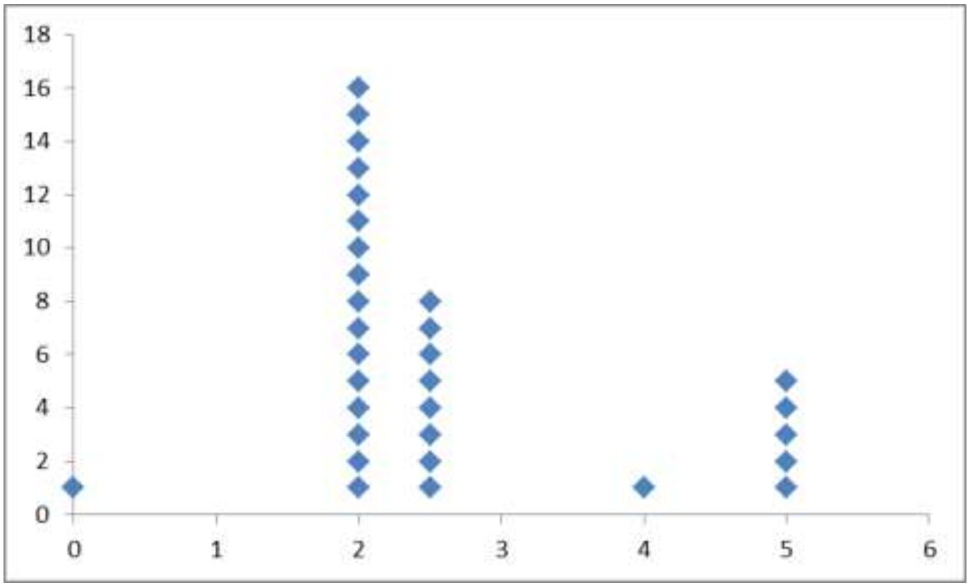


3) Give a matrix with the following null space: (5 points)

$$\text{span}\left(\left\{\begin{bmatrix}1\\6\\2\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}\right)$$

There are many possible answers. The simplest is probably:

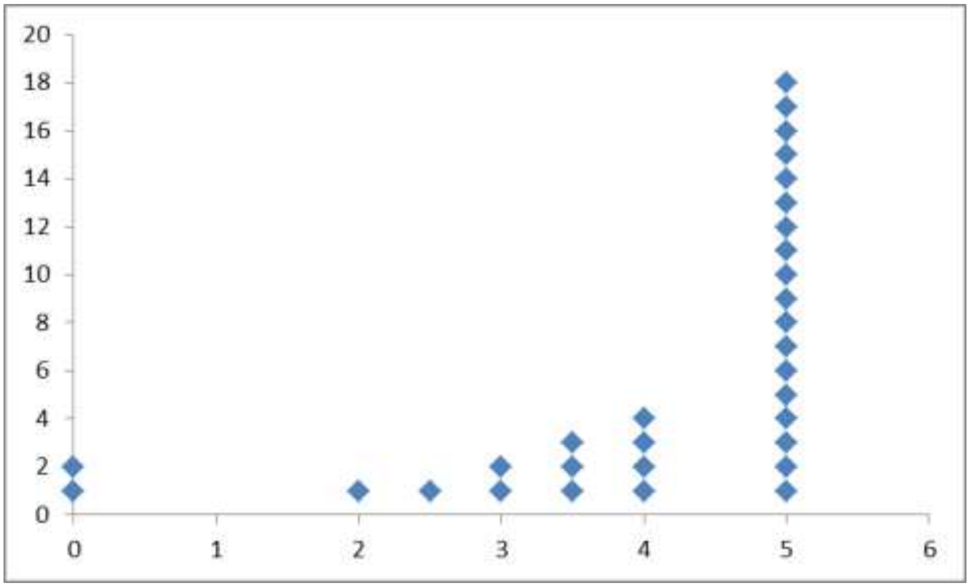
$$\begin{bmatrix}6 & -1 & 0\end{bmatrix}$$



4) Give an example of a homogeneous system of equations in which the associated linear transformation has nontrivial kernel. (5 points)

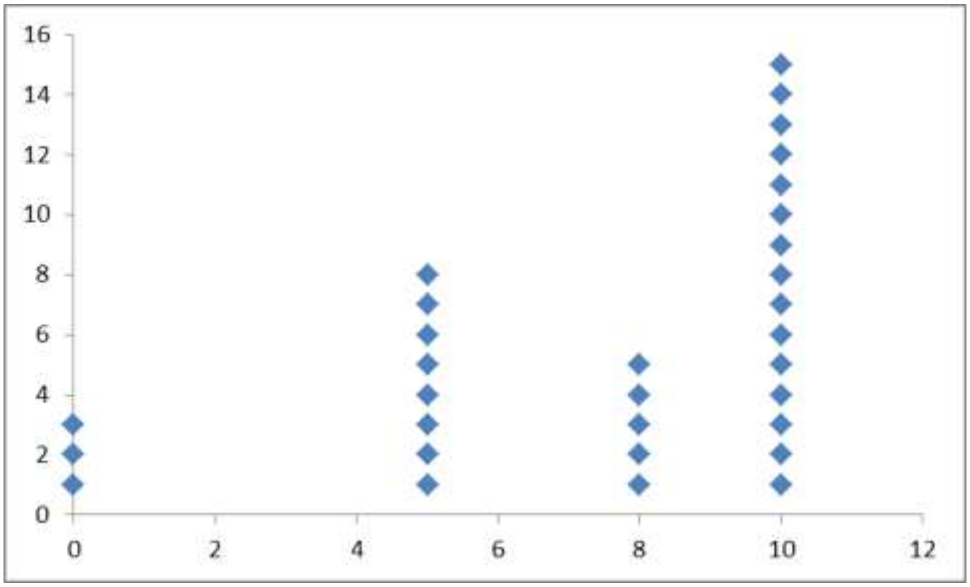
There are many possible answers. One simple such answer is:

$$x_1 + x_2 = 0$$



5) Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a linear operator with trivial kernel. Prove that the columns of $[T]$ are linearly independent. (10 points)

Denote the columns of $[T]$ as $\vec{v}_1, \dots, \vec{v}_n$. Assume $a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = 0$. Then as a matrix equation this means $[T]\vec{a} = \vec{0}$ where $\vec{a} = [a_1 \quad a_2 \quad \cdots \quad a_n]^t$. Because the kernel is trivial, $\vec{a} = \vec{0}$. That is to say, $a_1 = 0, a_2 = 0, \dots, a_n = 0$. Hence $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent. □



Grading:

10 points if it looks like you know what you're doing.

8 points if you seem to have an idea, but made some glaring mistakes.

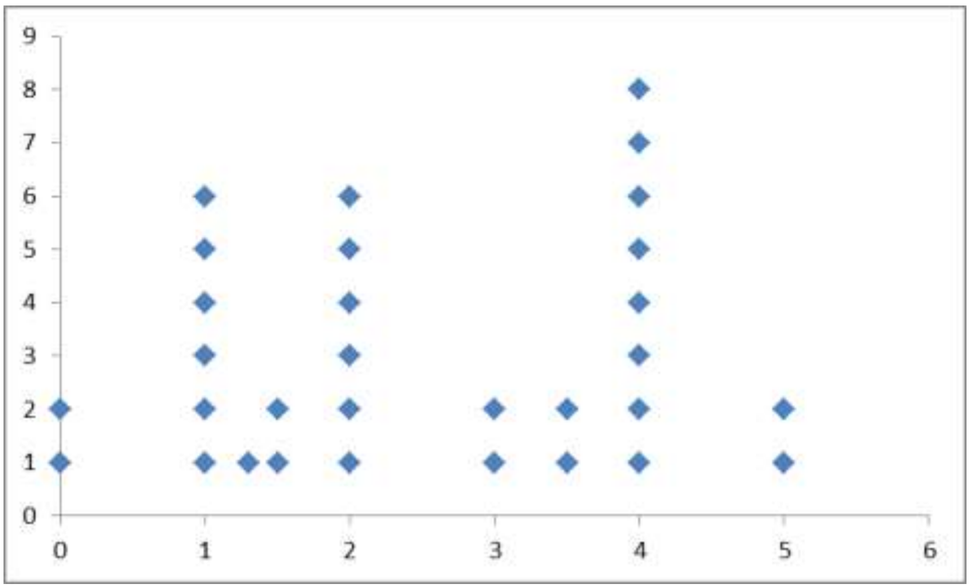
5 points if you're completely off base but made a reasonable attempt.

…a couple people appear to have memorized a proof for something else. Bad!

6) Find a basis for $\left\{\begin{bmatrix} 0 \\ 1 \\ a \\ a+b \\ c \end{bmatrix} : a, b, c \in \mathbb{R}\right\}$. (5 points)

There are many possible answers. The simplest is probably:

$$\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$



Note that technically speaking this set is not a subspace of $\mathbb{R}^n$; it was graded according to whether or not your answer was linearly independent and spanned this set.

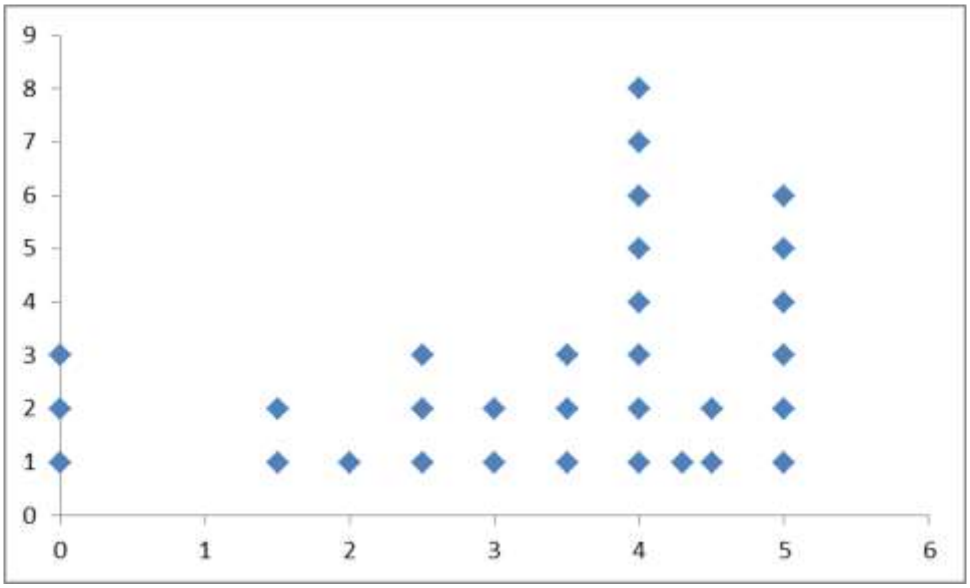
Note that this problem was re-graded several times. The pink marks are corrections to the red marks, and the orange marks are corrections to the red and/or pink marks.

7) Give an example of a linear operator T such that the associated linear system $[T]\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^3$. (5 points)

There are many answers. One possible answer is:

$$T: \mathbb{R}^3 \to \mathbb{R}^3$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

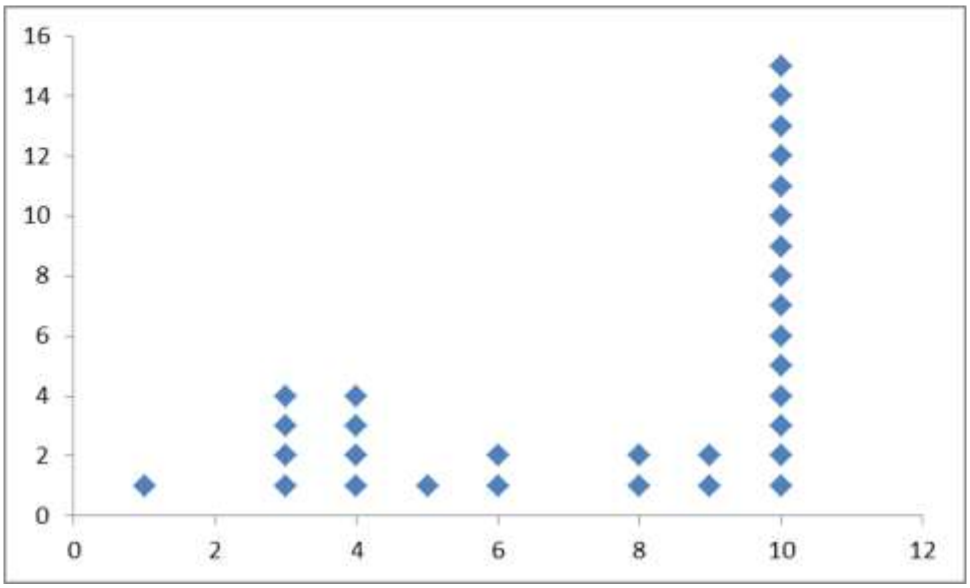


8) Find the null space of $\begin{bmatrix} 1 & -5 \\ -3 & 15 \\ 2 & -10 \end{bmatrix}$. (10 points)

$$\begin{bmatrix} 1 & -5 \\ -3 & 15 \\ 2 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solving $\begin{bmatrix} 1 & -5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, we find that x_2 can be free, while x_1 is five times x_2. Hence the null space is:

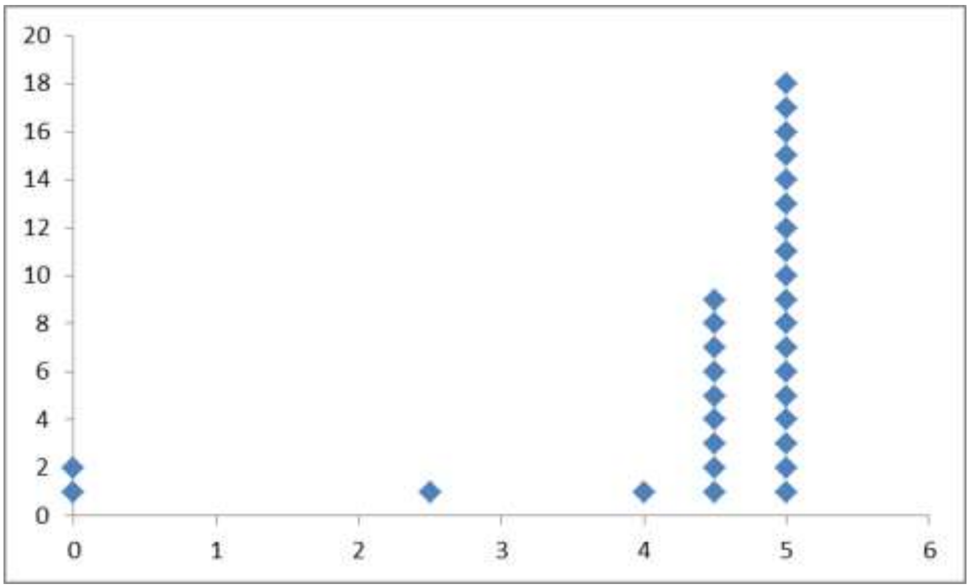
$$\text{span}\left(\left\{\begin{bmatrix} 5 \\ 1 \end{bmatrix}\right\}\right)$$



9) Alice is an aspiring linear algebraist. Find a set of three vectors in $\mathbb{R}^4$ such that when any one is removed, Alice can find two new vectors to add to the set to make a basis for $\mathbb{R}^4$. (5 points)

There are many possible answers. One such answer is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

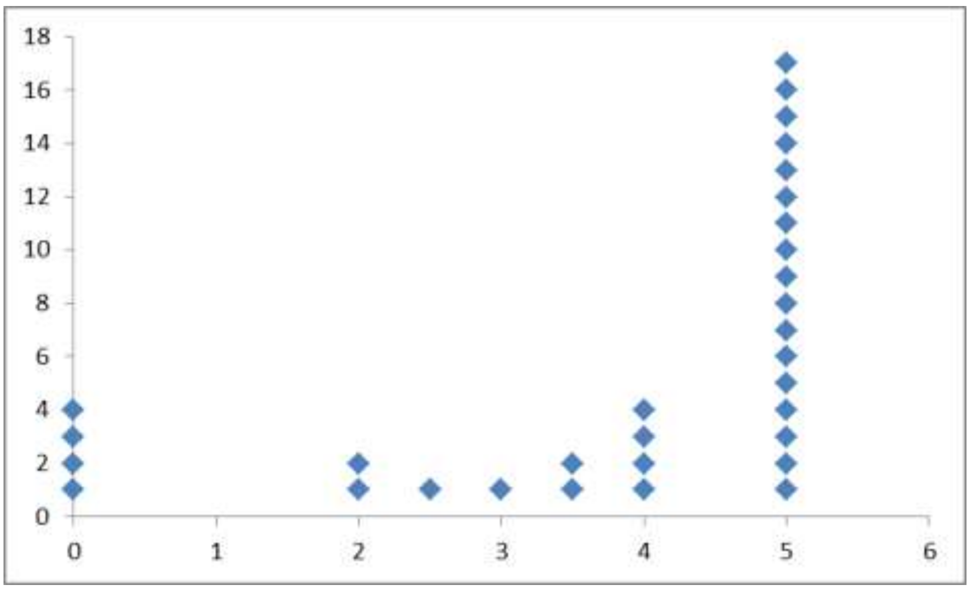


10) If $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$, find $[T^{-1}]$. That is, find the associated matrix of the inverse function. (5 points)

$$[T] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & : & 1 & 0 \\ 0 & 1 & : & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & : & 1 & -1 \\ 0 & 1 & : & 0 & 1 \end{bmatrix}$$

$$[T^{-1}] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$



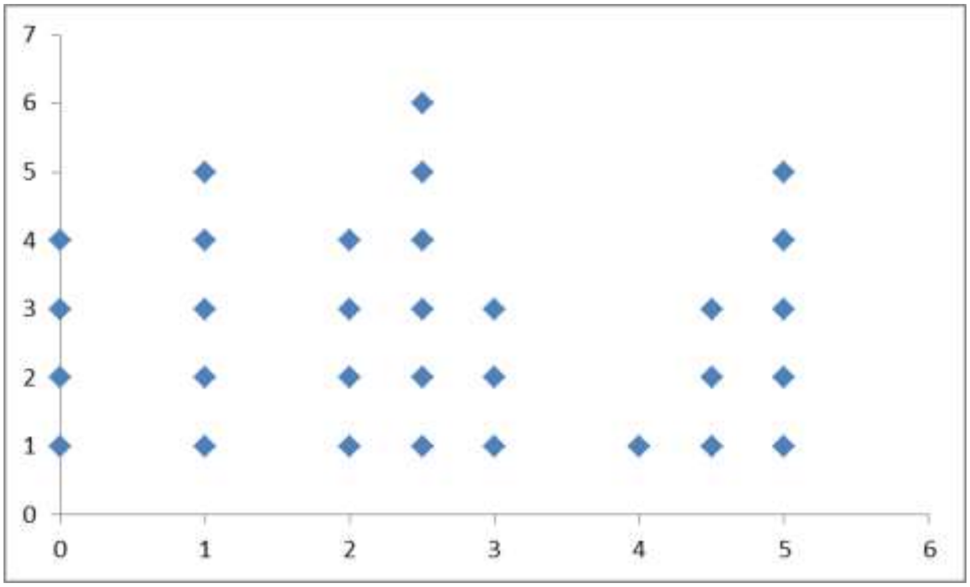
11) A 7×5 matrix has just 3 linearly dependent rows. How many free variables are there in the associated system of homogenous equations? (10 points)

This question was thrown out because it is not well defined: It is not clear what "having 3 linearly dependent rows" means. The question was intended to say "A 7×5 matrix has just 3 linearly independent rows."

In that case, the nullity of the matrix would be 2, so there would be two free variables.

12) Express 432 different bases for $\mathbb{R}^4$. (Hint: Don't try to write them all down) (5 points)

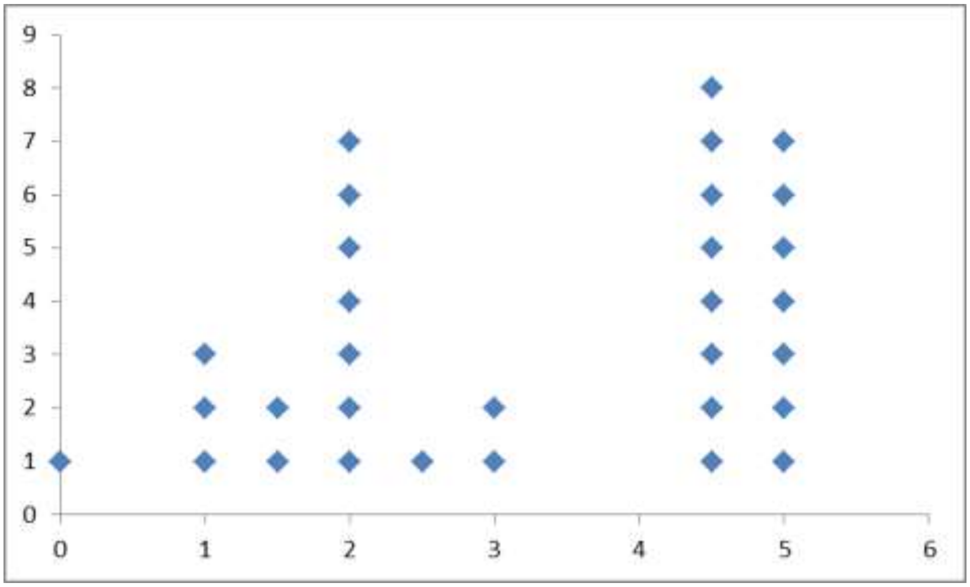
$$\left\{\begin{bmatrix}a\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\a\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\a\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\a\end{bmatrix}\right\} \text{ for } a = 1, \dots, 432$$



13) Find two examples of dimension 2 spaces. (5 points)

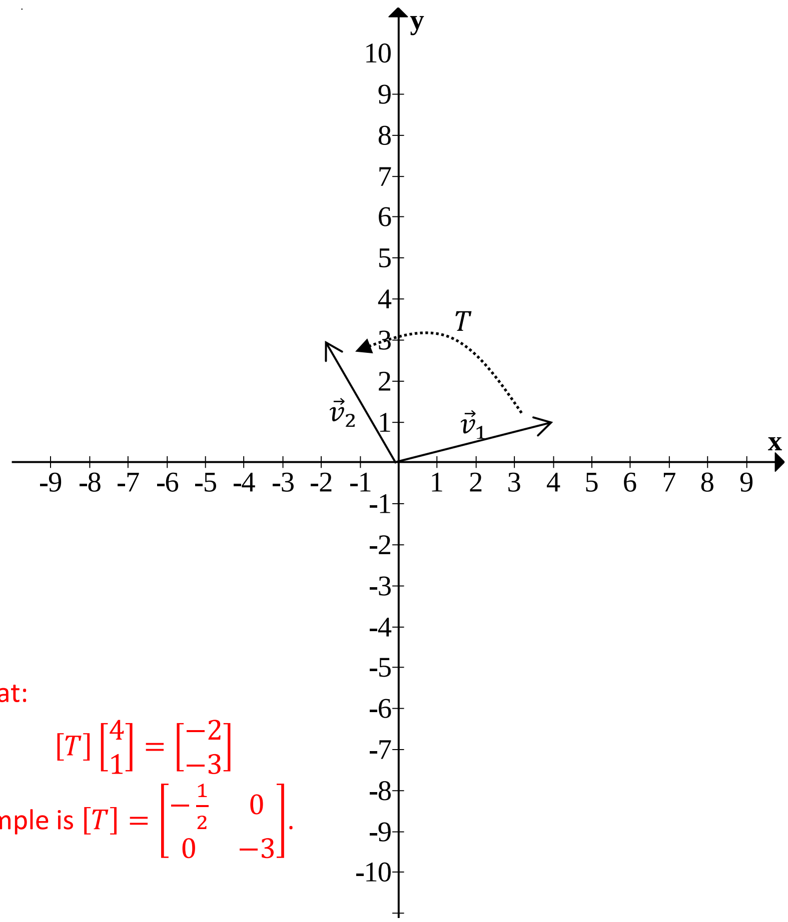
There are many such spaces. Two of these spaces are:

$\mathbb{R}^2$ and $\text{span}\left(\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}\right)$



Grading note: Points *were* deducted if you gave the same space written in two different ways, such as $\mathbb{R}^2$ and $\text{span}\left(\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\17\end{bmatrix}\right\}\right)$

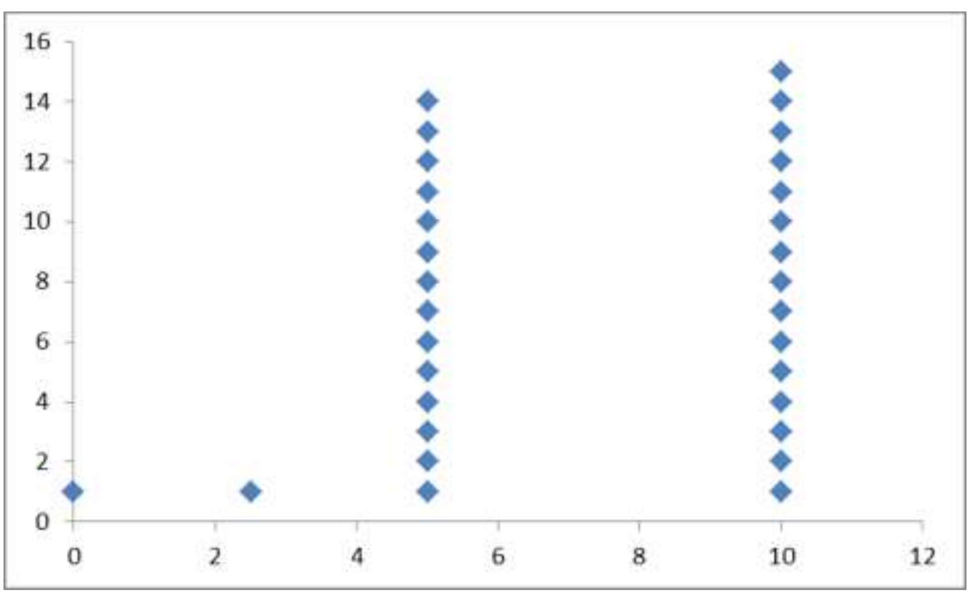
14) The graph below illustrates two vectors, $\vec{v}_1$ and $\vec{v}_2$. The picture illustrates that $T(\vec{v}_1) = \vec{v}_2$. Find the associated matrix$[T]$. (10 points)



We must find a 2×2 matrix $[T]$ such that:

$$[T]\begin{bmatrix}4\\1\end{bmatrix} = \begin{bmatrix}-2\\-3\end{bmatrix}$$

There are many such matrices. One example is $[T] = \begin{bmatrix}-\frac{1}{2} & 0\\ 0 & -3\end{bmatrix}$.



15) Find all values of x so that $\text{rank}(A) = 2$, when $A = \begin{bmatrix} -1 & 2 & 1 \\ 3 & 1 & 11 \\ 4 & 3 & x \end{bmatrix}$. (10 points)

The first two rows are linearly independent. Hence to be rank 2, the third row must be linearly dependent. Reducing A we find:

$$\begin{bmatrix} -1 & 2 & 1 \\ 3 & 1 & 11 \\ 4 & 3 & x \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & x-18 \end{bmatrix}$$

Hence for this matrix to have rank 2, x must be 18.