Name__Solutions____ Linear Algebra; Test 3

Throughout the test simplify all answers except where stated otherwise.

1) Find the following: (10 points)

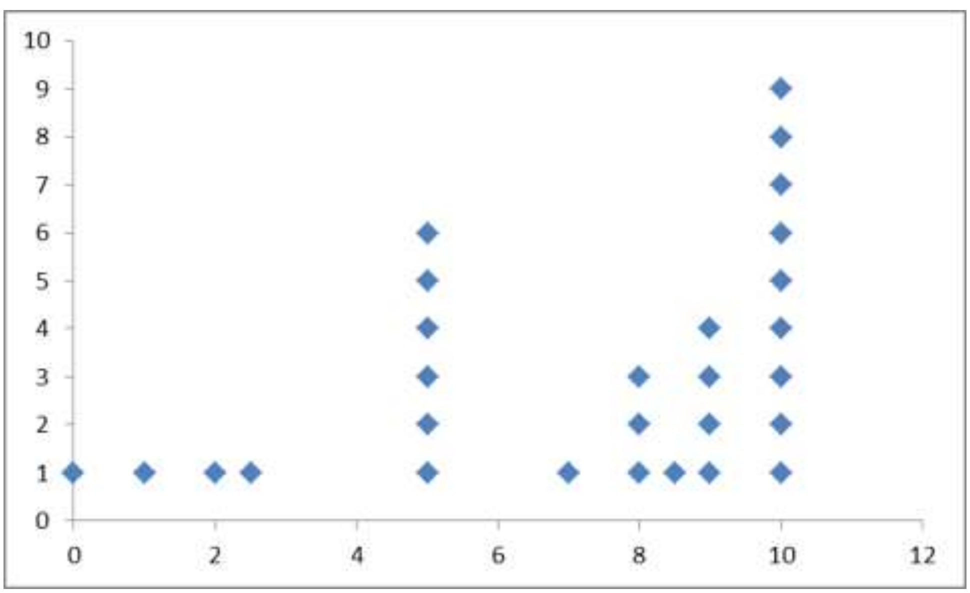
$$\begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1/6 & 0 & 0 & 1/2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1/6 & 0 & 0 & 1/2 \end{vmatrix} = 4 \cdot \begin{vmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} \end{vmatrix} = 4 \cdot 2 \cdot \begin{vmatrix} 1 & 3 \\ \frac{1}{6} & \frac{1}{2} \end{vmatrix} = 4 \cdot 2 \cdot \left(\frac{1}{2} - \frac{1}{2}\right) = 0$$

Or note that $R_4 = \frac{1}{6} R_1$ so the rows are linearly independent, so the matrix must have determinant 0.

Or $$\begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1/6 & 0 & 0 & 1/2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} - 3 \cdot \begin{vmatrix} 0 & 4 & 0 \\ 0 & 0 & 2 \\ \frac{1}{6} & 0 & 0 \end{vmatrix} = 1 \cdot 4 \cdot 2 \cdot \frac{1}{2} - 3 \cdot \frac{1}{6} \cdot \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} = 4 - 4 = 0$$

Note that there are some tricks that work for 2x2 and 3x3 matrices that do not work in general. There are also methods of calculating the determinant that do generalize to any size. If you use a method we didn't learn in class, be sure to read up on it to make sure you know when, why, and how it works.



2) Find the eigenvalues and eigenvectors of the following matrix: (10 points)

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

Characteristic equation:

$$\begin{vmatrix} 2-x & 1 \\ -1 & -x \end{vmatrix} = 0$$
$$(2-x)(-x) + 1 = 0$$
$$x^2 - 2x + 1 = 0$$
$$(x-1)^2 = 0$$

The eigenvalues are both 1:
$\lambda_1 = 1, \lambda_2 = 1.$
Note that there are still two eigenvalues, counting multiplicity. Just saying "1" is not quite correct unless you note that it has multiplicity 2.

Plug λ back in to find the eigenvectors: the null space of $\begin{bmatrix} 2-1 & 1 \\ -1 & -1 \end{bmatrix}$

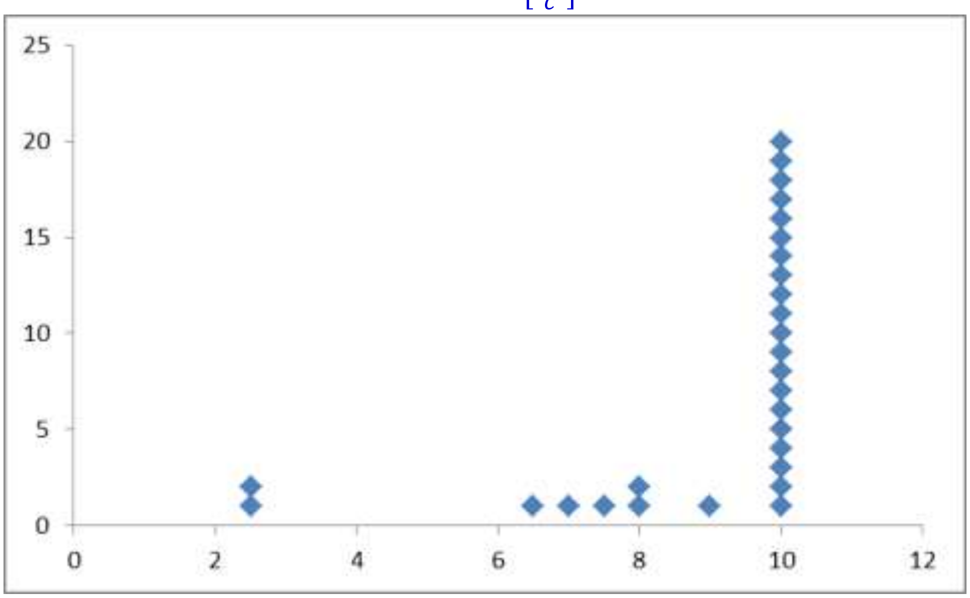
As we expect, this matrix has dependent rows; it is equivalent to:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

which has null space span $\left(\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}\right)$.

Hence one eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(If that one vector is all you put, I'm assuming you know that $\begin{bmatrix} -c \\ c \end{bmatrix}$ is also an eigenvector for any number c.)



3) Find a matrix P and diagonal matrix D such that $A = PDP^{-1}$, where A is the matrix below. (20 points)

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

We know that D consists of the eigenvalues and P the eigenvectors. Hence we must find those.

$$\begin{vmatrix} 2-x & 0 & 0 \\ -1 & -x & 1 \\ 0 & 0 & 1-x \end{vmatrix} = (2-x)\begin{vmatrix} -x & 1 \\ 0 & 1-x \end{vmatrix} = (2-x)(-x)(1-x) = 0$$

$$\lambda_1 = 2, \lambda_2 = 0, \lambda_3 = 1$$

$\lambda_1 = 1$:

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda_2 = 0$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda_3 = 1$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

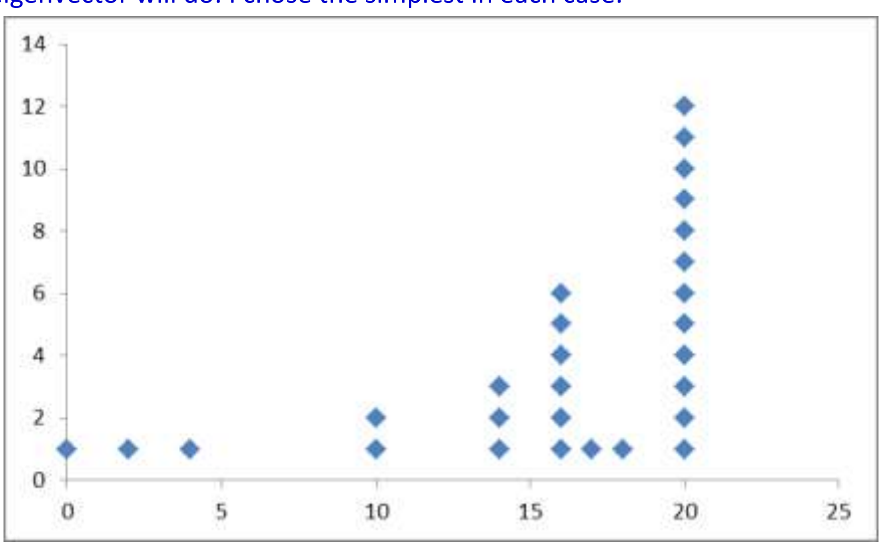
$$v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Hence:

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that you can change the order of the columns as long as you do so in both matrices.

Note that any eigenvector will do: I chose the simplest in each case.



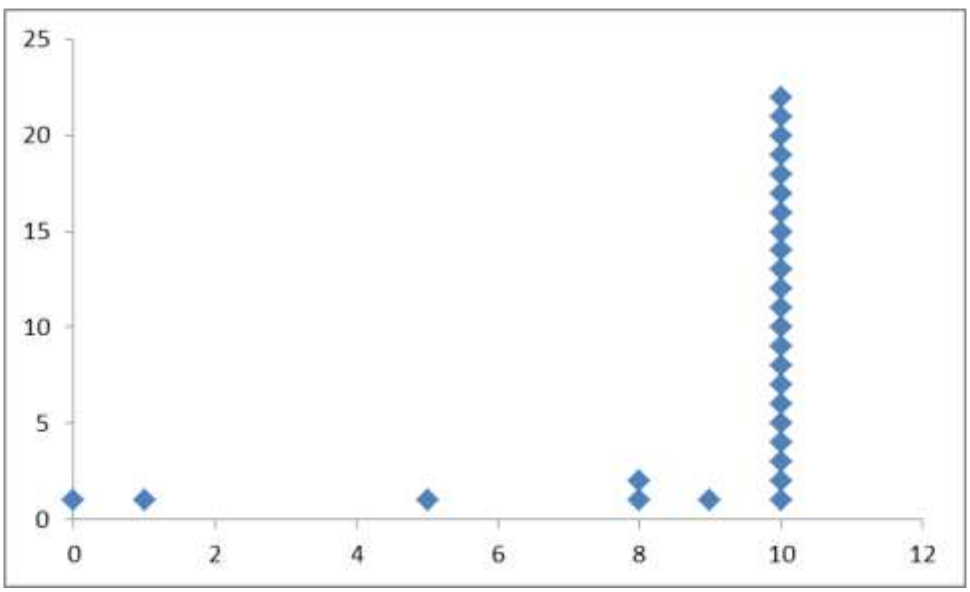
Let $\beta_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ 4 \\ 3 \end{bmatrix} \right\}$, and $\beta_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}$.

4) Write $\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}_{\beta_1}$ in terms of the standard basis. (10 points)

$$[I_4]_{\beta_1}^{S} = \begin{bmatrix} 0 & 0 & 0 & 7 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}_{\beta_1} = \begin{bmatrix} 0 & 0 & 0 & 7 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 \\ 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 7 \\ 5 \end{bmatrix}$$

Note that all unlabeled vectors are representations in the standard basis.
Note that in class we never put subscripts or superscripts on matrices. The notation with $[I_4]_a^b$ is meant to be illustrative, but to really see what's happening here you need to think of everything in terms of linear transformations instead of matrices. I suggest using it to help sort through the spaces but to leave it at that.

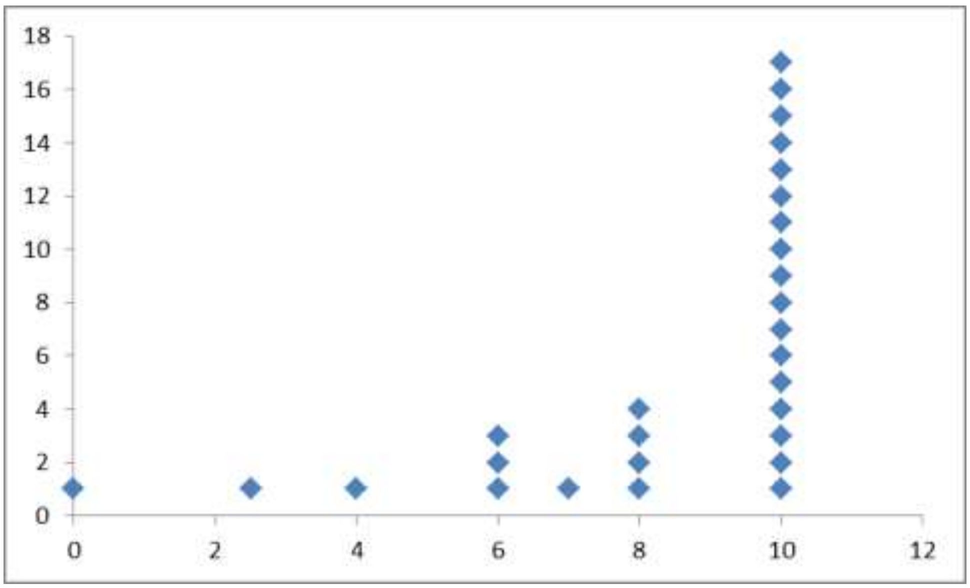


5) Write $\begin{bmatrix}5\\6\\7\\8\end{bmatrix}_S$ in terms of β_1. (10 points)

(Do not simplify your answer: any mathematical expression that works out to the correct answer is acceptable.)

$$\begin{bmatrix}0 & 0 & 0 & 7\\1 & 0 & 0 & 2\\0 & 1 & 0 & 4\\0 & 0 & 1 & 3\end{bmatrix}^{-1}\begin{bmatrix}5\\6\\7\\8\end{bmatrix}$$

If you calculated it, the inverse here isn't that bad, but does require a few steps to work out.



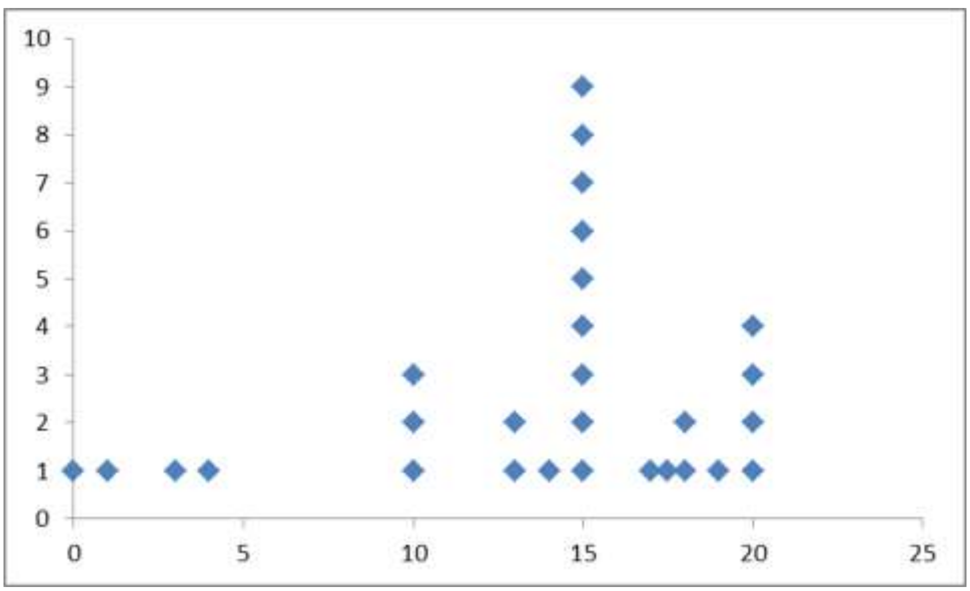
6) Find $[I_4]_{\beta_1}^{\beta_2}$, the change of basis matrix from β_1 to β_2. Be sure to show all your work. (20 points)

$$[I_4]_{\beta_1}^{\beta_2} = [I_4]_S^{\beta_2}[I_4]_{\beta_1}^S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 & 7 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 7 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 7 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

The inverse only requires one step to work out (Again, be careful of tricks that don't generalize!)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & : & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & : & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & : & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & : & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & : & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & : & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$



7) Give an example of 2×2 matrix that is not diagonalizable. (5 points)

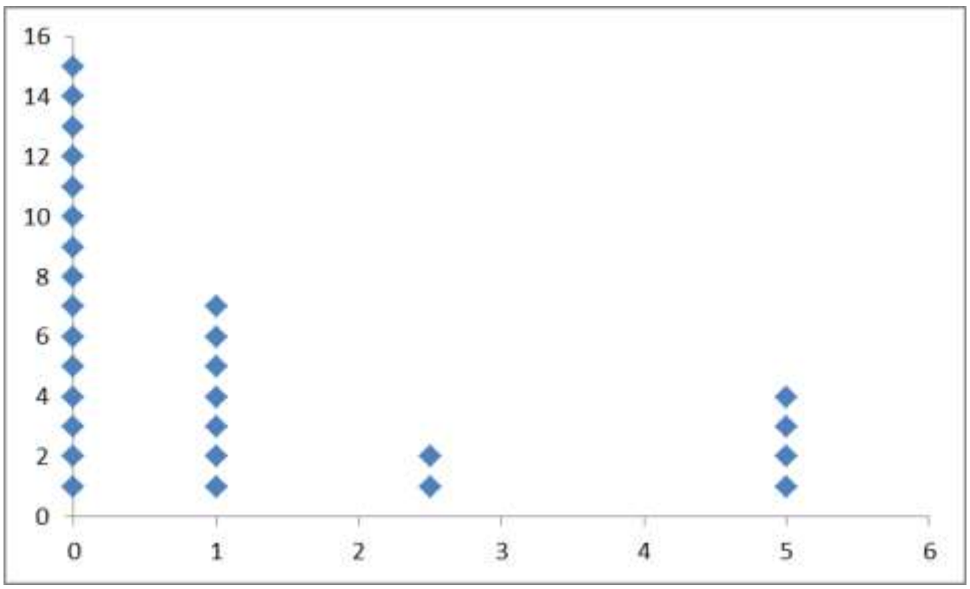
Here we need a matrix that only has one linearly independent eigenvector. We know eigenvectors from different eigenvectors are linearly independent, so we must have a repeated eigenvalue.

Then once we have a repeated eigenvalue, we need to make sure the null space has dimension one.

Looking back at problem #2, we have a perfect example!

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

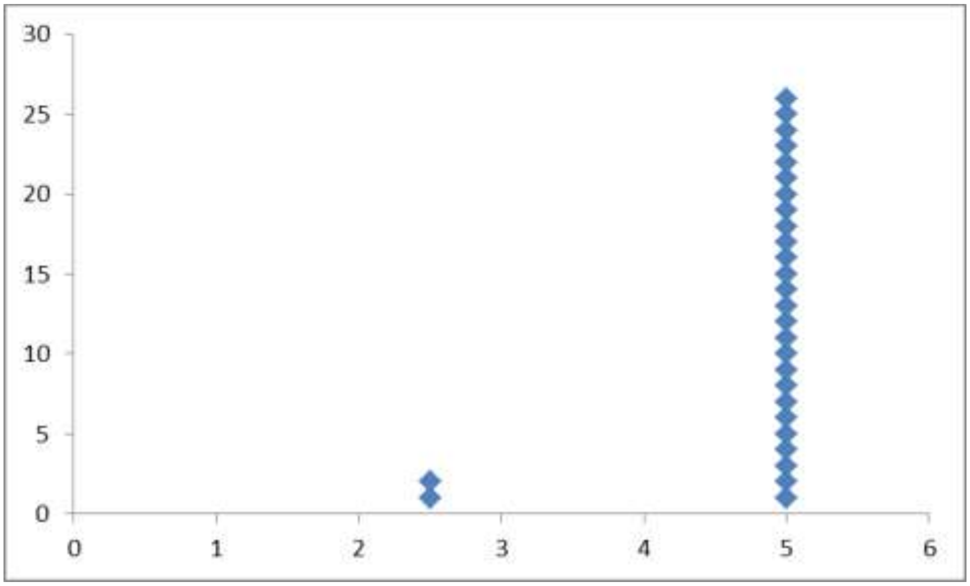
Note in particular that we don't care whether or not the matrix is invertible. That is a very different property from diagonalizable. In fact, "diagonalizable" straddles several of our nice equivalence classes!



8) Give an example of a 3×3 matrix that has determinant 42π. (5 points)

A diagonal matrix is the simplest, but there are certainly others.

$$\begin{bmatrix} 42\pi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

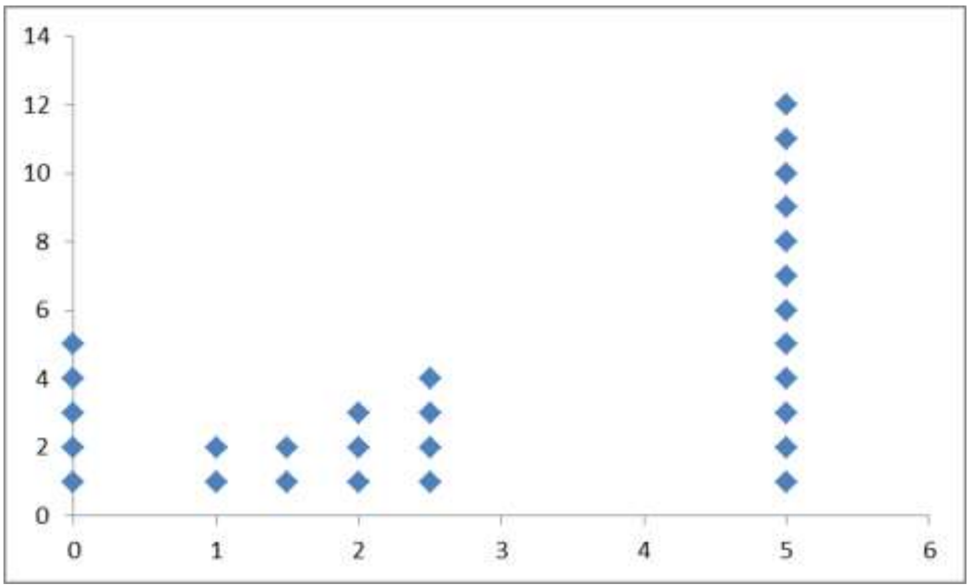


9) Suppose A is a 4×4 matrix with eigenvalues $5, 6$, and 7, with 7 having multiplicity 2. If A is not diagonalizable, what is the rank of $A - 7I_4$? (5 points. Provide an explanation for partial credit; otherwise all or nothing)

Note that A must not be diagonalizable, meaning it must not have a full set of 4 eigenvectors. It already has at least three (one from each eigenvalue), so we don't have much to work with!

The eigenvalue 7 is repeated, however. Hence there can be either 1, or 2 linearly independent eigenvectors with that eigenvalue. That is the homogenous system associated to $A - 7I_4$ can have either 1 or 2 free variables. (Each free variable gives an eigenvector linearly independent from the rest).

So that system of equations must have just 1 free variable, meaning $A - 7I_4$ must have rank 3.



10) Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. It is known that two eigenvectors of A are $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find the following: (5 points)

$$\begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}^5 \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}$$

Because those are eigenvectors, we know that:

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$$

Hence:

$$\begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}^5 \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \right)^5 \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^5 = \begin{bmatrix} \lambda_1^5 & 0 \\ 0 & \lambda_2^5 \end{bmatrix}$$

(Remember matrix multiplication is not commutative. So I had to rig this problem specially so that everything cancelled)

Working out the eigenvalues, we find that:

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = - \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence

$$\begin{bmatrix} \lambda_1^5 & 0 \\ 0 & \lambda_2^5 \end{bmatrix} = \begin{bmatrix} (-1)^5 & 0 \\ 0 & 4^5 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1024 \end{bmatrix}$$